Long-Range Forces and Neutrino Mass

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Abstract

We explore the limits on neutrino mass which follow from a study of the long-range forces that arise from the exchange of massless or ultra-light neutrinos. Although the 2-body neutrino-exchange force is unobservably small, the many-body force can generate a very large energy density in neutron stars and white dwarfs. We discuss the novel features of neutrino-exchange forces which lead to large many-body effects, and present the formalism that allows these effects to be calculated explicitly in the Standard Model. After considering, and excluding, several possibilities for avoiding the unphysically large contributions from the exchange of massless neutrinos, we develop a formalism to describe the exchange of massive neutrinos. It is shown that the stability of both neutrons stars and white dwarfs in the presence of many-body neutrino-exchange forces implies a lower bound, $m \gtrsim 0.4\ \mathrm{eV}/c^2$ on the mass m of any neutrino.

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I. INTRODUCTION AND SUMMARY

There is considerable theoretical and experimental interest at present in the question of whether neutrinos have mass [1–4]. On the theoretical side the existence of massive neutrinos would require a modification of the standard SU(2) $\times$ U(1) model of the electroweak interaction, and could also provide some insight into the question of how other particles acquire their masses. These and other related theoretical questions, such as the missing mass problem in the Universe [1,5], have served to stimulate continuing experimental searches for evidence of a nonzero neutrino mass. To date there is no compelling direct evidence for massive neutrinos, and the existing upper limits for ν_e, ν_μ, and ν_τ (or for the corresponding antiparticles) are [6]

$$m(\nu_e) \lesssim 7.0 \text{ eV}; \qquad m(\nu_\mu) \lesssim 0.27 \text{ MeV}; \qquad m(\nu_\tau) \lesssim 31 \text{ MeV}. \tag{1.1}$$

Notwithstanding the bounds in Eq. (1.1), there is mounting indirect evidence for massive neutrinos from a number of different experiments as discussed in Refs. [7–11]. We will return to Eq. (1.1) in Sec. VIII below.

The object of the present paper is to develop a new technique for studying neutrino mass which exploits the fact that massless neutrinos give rise to long-range forces. Recent interest in the question of weak long-range forces [12–14] has focussed on the possible existence of new ultra-light bosonic fields as the mediators of such interactions. It is well known, however, that long-range forces can also arise from the exchange of neutrino-antineutrino ($\nu\bar{\nu}$) pairs, and the history of various attempts to calculate the $\nu\bar{\nu}$-exchange force is summarized in Ref. [15]. The first correct calculation of the 2-body $\nu\bar{\nu}$-exchange potential was carried out by Feinberg and Sucher (FS) [15] who used an effective low-energy 4-fermion interaction involving only charged currents. They found for the potential energy $V_{ee}^{(2)}(r)$ describing the interaction of two electrons via the $\nu\bar{\nu}$-exchange diagram shown in Fig. 1,

$$V_{ee}^{(2)}(r) = G_F^2/4\pi^3 r^5, \tag{1.2}$$

where $r = |\vec{r}_1 - \vec{r}_2|$ is the separation of the electrons, and $G_F = 1.16639(2) \times 10^{-5}$ GeV$^{-2}(\hbar c)^3$

is the Fermi decay constant. Subsequently Feinberg, Sucher, and Au (FSA) [16] recalculated $V_{ee}^{(2)}(r)$ in the framework of the Standard Model and obtained

$$V_{ee}^{(2)}(r) = G_F^2(2\sin^2\theta_W + 1/2)^2/4\pi^3 r^5, \tag{1.3}$$

where θ_W is the weak mixing angle, with $\sin^2\theta_W = 0.2319(5)$. The result in Eq. (1.3) has been rederived recently by Hsu and Sikivie [17] using a different formalism from that of FS. Numerically, $V_{ee}^{(2)}(r) = 3 \times 10^{-82} r^{-5}$ eV where r is in meters, from which one can see that the interaction energy of two electrons arising from neutrino-exchange is smaller than their mutual gravitational interaction for $r \gtrsim r_0 = 5 \times 10^{-8}$m. However, for $r \lesssim r_0$, the electroweak interaction arising from γ- and Z^0-exchange would dominate, and hence there appears to be no distance scale over which $V_{ee}^{(2)}(r)$ leads to detectable effects. As we note in Appendix A, the neutrino-neutron and neutrino-proton coupling constants, a_n and a_p respectively, are even smaller than the neutrino-electron coupling constant a_e, so the same conclusions would hold for the corresponding 2-body potentials $V_{nn}^{(2)}(r)$ and $V_{pp}^{(2)}(r)$. In what follows it will be shown that although the 2-body potential energy is indeed extremely small, the many-body interaction energy arising from neutrino-exchange can be extremely large in neutron stars and white dwarfs. This observation eventually leads to the conclusion that there is a lower bound on the mass of any neutrino or antineutrino, as given in Eq. (8.12) below.

Many-body effects arising from neutrino-exchange have been considered previously by a number of authors [18–20], often in connection with attempts to understand the gravitational interaction. Feynman [20] considered the many-body interaction that would describe the effective 2-body force that arises when two test masses interact via neutrino-exchange in the presence of distant matter. Although Feynman's attempt to explain the inverse-square law of Newtonian gravity in terms of neutrino-exchange ultimately proved unsuccessful, he made an observation about such interactions which forms part of the basis of the present work. Feynman noted that higher order (in G_F) many-body interactions could be important because they depended on higher powers of the masses of the interacting objects. This is

a special case of an observation made earlier by Primakoff and Holstein (PH) [21] in their classic analysis of many-body effects in atomic and nuclear systems. PH noted that in a nucleus containing N particles the magnitude of the total k-body interaction ($k = 2, 3, \cdots$). grows as the binomial coefficient $\binom{N}{k}$,

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}, \tag{1.4}$$

which counts the number of distinct k-body amplitudes that can be formed from N particles. Subsequently Chanmugam and Schweber [22] suggested that many-body electromagnetic effects could be important in white dwarfs due to the presence of the binomial coefficient. We will return shortly to discuss the combinatorics of many-body diagrams using Eq. (1.4). Subsequently Hartle [18] used the same phenomenological weak coupling employed by FS to evaluate the $\mathcal{O}(G_F^4)$ 4-body contribution corresponding to the diagram in Fig. 2a and obtained the result quoted in Eq. (3.50) below. Using this result Hartle demonstrated that the energy for two electrons interacting with each other and with a spherical shell of matter varied with the separation r of the electrons approximately as $1/r$. For the systems that Hartle considered the effects arising from the 4-body potential were too small to be of interest. Moreover, even though electrons can interact via long-range neutrino-exchange forces, these cannot be used to measure the electron-number of a black hole ("a black hole has no neutrino hair" [23]). Although these results appear to support the conclusion derived from the 2-body potential that neutrino-exchange forces are unimportant, we show in what follows that this is not necessarily the case when computing the self-energy of a compact object such as a neutron star or a white dwarf. In fact the self-energy of such an object arising from the exchange of massless neutrinos can be catastrophically large, and this result eventually leads to the conclusion that neutrinos cannot be massless.

Since the large neutrino-exchange energy-density in a neutron star arises from many-body interactions among neutrons, it is appropriate to ask why many-body effects play so important a role here, but are relatively unimportant in most other circumstances. The explanation can be found in Table I which compares neutrino-exchange to other known forces

with respect to three conditions which determine when many-body effects become significant. In order for there to be an enhancement effect arising from the binomial coefficient $\binom{N}{k}$, it is necessary in the present context to have a large number of particles interacting with sufficient strength in a small volume.

The first condition is that the force be long-range, which is what allows a given particle to interact with many other particles in the first place. We see from Table I that both the strong interaction and the weak interaction mediated by Z^0-exchange fail to meet this condition. In the case of strong interactions it is well known that the forces among nucleons saturate, i.e., that each nucleon in a nucleus interacts with only a limited number of other nucleons [24,25]. This is evidenced by the fact that the strong interaction binding energy of a nucleus does not increase with baryon number A as $\binom{A}{2} = A(A-1)/2$, as might be expected from a 2-body interaction, but grows approximately as A for $A > 4$. In nuclei this can be understood in terms of exchange forces and the existence of a repulsive hard core, but for a macroscopic object the finite range of nuclear forces is also significant.

The second condition in Table I is that the force couple to a charge capable of having a large expectation value in a macroscopic system such as a neutron star. Evidently a long-range force will not produce a significant many-body interaction unless the constituents that are capable of interacting with one another have a (large) net value of the appropriate charge and a non-zero charge density. For the electromagnetic interaction a number of independent arguments show that the net charge Z of a typical neutron star is less than $10^{36}e_0$, where e_0 is the charge of the electron, and may be as small as $10^{17}e_0$ [26]. The latter estimate is smaller than N by a factor of order 10^{-40}, which by itself would be sufficient to suppress many-body effects. However, there are in addition the suppression effects originally discussed by Primakoff and Holstein [21] and these, along with the small net Z, suggest that many-body electromagnetic interactions should be small in neutron stars.

The last condition in Table I requires that the interaction be sufficiently strong so that, when many-body interactions are in fact possible, their net effect be large. This condition can be understood with reference to the gravitational interaction as follows: We note that the

quantity which determines when gravity is sufficiently strong that many-body interactions would be important is the dimensionless potential Φ,

$$\Phi = \frac{G_N M}{R c^2}, \tag{1.5}$$

where G_N is the Newtonian gravitational constant, M is the mass of the neutron star, and R is its radius. Using the numerical results in Sec. V below we find that $\Phi \simeq 0.2$ for the binary pulsar PSR 1913+16, which is the "typical" neutron star we are considering. The fact that Φ must be small is obvious since $\Phi \geq 1/2$ would correspond to a black hole. It follows that for ordinary matter many-body gravitational effects will never dominate over the two-body interaction, although they may lead to detectable effects in some systems [27]. Another way of understanding this result is to note that in natural units, where G_F and G_N have the same dimensions,

$$G_N/G_F \simeq 10^{-33}. \tag{1.6}$$

If follows from Eq. (1.6) that if one were to reduce G_F by a factor of order 10^{33} so as to make the weak neutrino-exchange interaction have a strength comparable to that of gravity, then many-body neutrino-exchange effects would become relatively unimportant, just as gravitational many-body effects are.

The preceding discussion can be summarized as follows: We see from Table I that each of the four known fundamental forces fails to satisfy at least one of the conditions that must be met for many-body effects to be important in a macroscopic system such as a neutron star. The only known interaction which is of long-range, and where there is a strong coupling to a charge for which a neutron star is non-neutral, is the force arising from neutrino exchange. It is for these reasons that many-body exchange effects can be significant for this interaction, while they are relatively unimportant for the others.

To understand quantitatively how many body neutrino effects can give rise to a catastrophically large energy-density, we follow the discussion of Feinberg and Sucher [15] who note that the functional form of $V_{ee}^{(2)}(r)$ can be inferred on dimensional grounds. Since

the Standard Model is renormalizable, the only dimensional factors upon which the static spin-independent potential can depend are G_F and r. Evidently the exchange of a single $\nu\overline{\nu}$ pair must be proportional to G_F^2, from which it follows that $V_{ee}^{(2)} \propto G_F^2/r^5$ in agreement with Eq. (3.19) below. An analogous argument shows that for $k \geq 3$ the k-body neutrino-exchange contribution $W^{(k)}$ to the binding energy of a neutron star must be proportional to $(G_F^k/R^{2k+1})\binom{N}{k}$, where R is the radius of the neutron star, which is assumed for present purposes to contain only neutrons and to have a uniform density. Since $\binom{N}{k} \simeq N^k/k!$ for $k \ll N$, it follows that

$$W^{(k)} \sim \frac{1}{k!}\,\frac{1}{R}\left(\frac{G_F N}{R^2}\right)^k . \tag{1.7}$$

For a typical neutron star $(G_F N/R^2) = \mathcal{O}(10^{13})$ [see Sec. V below], and hence it follows from Eq. (1.7) that for $k \ll N$ higher order many-body interactions make increasingly larger contributions to $W^{(k)}$. It can be shown that $W^{(8)}$ would exceed the known mass-energy of a typical neutron star, and that $W = \sum_k W^{(k)}$ can exceed the total mass-energy of the Universe, as we discuss below.

The calculation of the self-energy of a neutron star or white dwarf arising from neutrino exchange closely parallels the calculation of the electrostatic energy of a spherical charge distribution such as a nucleus arising from photon exchange. For this reason, we begin in Sec. II by reviewing the derivation of the familiar electrostatic (Coulomb) result, which is given in Eq. (2.1). For present purposes the electrostatic derivation can be viewed as proceeding in three steps: 1) First the 2-body potential is derived from covariant perturbation theory. 2) The 2-body potential is then integrated over a spherical volume to obtain the contribution from a single pair of charges. 3) The result of the preceding calculation is then multiplied by the factor $\binom{N}{2} = N(N-1)/2$ which represents the number of pairs that can be formed from N charges. Particular attention is devoted to the question of integrating the 2-body Coulomb potential over a spherical volume, which is carried out in two different ways. One method uses a geometric probability technique, which is later generalized to apply to the many-body case.

Sec. III presents the derivation of the k-body potential $V^{(k)}$ arising from neutrino-exchange, using a formalism developed by Schwinger [28] and originally applied to this problem by Hartle [18]. After reproducing the 2-body result of FS and the 4-body potential obtained by Hartle, the 6-body potential is derived [Eq. (3.53)] and then used to generalize to the k-body case [Eqs. (3.55)–(3.58)]. In Sec. IV we discuss the integration of the k-body results over a spherical volume of radius R, which leads eventually to Eq. (4.22). When this expression is multiplied by $\binom{N}{k}$ the result is the net k-body contribution to the self-energy of the object in question. After the sum over k is carried out, the self-energy W of a spherical collection of N neutrons can be approximated by the expression in Eq. (5.23). We then proceed to demonstrate that, barring accidental cancellations, $W/M \gg 1$ for compact objects such as neutron stars and white dwarfs, where M is the (known) mass of each object. After considering in Sec. VI, and excluding, several alternative explanations for the unphysically large value of W/M, we turn in Sec. VII to recalculate W when the exchanged neutrinos have a nonzero mass m. As expected, the expression for W acquires additional factors proportional to $\exp(-mR)$ which suppress the neutrino-exchange contribution. One can then calculate the minimum value of m required to reduce W to a physically acceptable value, and the result is presented in Sec. VIII. From Eq. (8.12) we find for m,

$$m \gtrsim 0.4 \text{ eV}/c^2, \tag{1.8}$$

which is consistent with the upper limits quoted in Eq. (1.1).

Appendix A contains a summary of our notation and metric conventions, as well as a discussion of the neutrino couplings in the Standard Model. In Appendix B we present Hartle's (unpublished) derivation [29] of the Schwinger formula for W, along with some additional technical details of its application to the present problem. Appendix C contains a detailed discussion of the geometric probability formalism used in integrating the many-body potentials over a spherical volume. In Appendix D we discuss the formalism for summing the many-body contributions, and in Appendix E we generalize the Schwinger-Hartle formalism to apply to massive neutrinos.

II. ELECTROSTATIC ENERGY OF A SPHERICAL CHARGE DISTRIBUTION

As noted in the Introduction, the present calculation of the energy-density of a spherical neutron star arising from neutrino-exchange parallels that of the electrostatic energy of a spherical charge distribution. For this reason we review the derivation of the familiar result for the Coulomb energy W_C of a spherical nucleus [30],

$$W_C = \frac{3}{5} Z(Z-1)\frac{e_0^2}{R}, \tag{2.1}$$

where R is the effective Coulomb radius, Ze_0 is the nuclear charge, and $e_0^2/\hbar c \simeq 1/137$. (We use e_0 to denote the charge of the electron to avoid confusion with the transcendental number e: $\ln e = 1$.) As noted in the Introduction, we can view the derivation of Eq. (2.1) as proceeding in three steps: (1) Determine the 2-body interaction $V_C(\vec{r}_{12})$ between two point charges; (2) calculate the interaction energy U_C between the two point charges which are assumed to be uniformly distributed inside a sphere of radius R; (3) generalize the 2-body result U_C to the Z-body electrostatic energy W_C by incorporating the appropriate combinatoric factors.

A. Calculation of the Two-Body Potential V_C

The electrostatic potential energy is dominated by the two-body contribution arising from the one-photon-exchange amplitude. In this case the evaluation of the expression for the potential energy $V_C(\vec{r}_{12})$ of two interacting particles is trivial, and the result is the familiar Coulomb potential

$$V_C(\vec{r}_{12}) = \frac{e_0^2}{r_{12}}, \tag{2.2}$$

where $r_{12} = |\vec{r}_1 - \vec{r}_2|$ is the distance between the charges. Although there are also in principle contributions from many-body electromagnetic interactions, these are small for reasons discussed originally by Primakoff and Holstein [21]. By contrast the weak energy arising from neutrino-exchange is dominated by many-body interactions, and hence the weak potential is correspondingly more complicated.

B. Integration Over a Sphere: U_C

This is the most difficult of the three steps in the weak-interaction case. For the Coulomb interaction in Eq. (2.2) we are interested in evaluating the energy of a single pair of charges 1 and 2 ($e_1 = e_2 \equiv e_0$), having a uniform probability distribution in a volume $(4/3)\pi R^3$, so that the effective number density ρ produced by each charge is

$$\rho_1 = \rho_2 \equiv \rho = \frac{1}{(4/3)\pi R^3}. \tag{2.3}$$

From Eq. (2.2), the potential energy arising from the Coulomb interaction between the charges $\rho_1 d^3x_1$ and $\rho_2 d^3x_2$ centered at $\vec{r}_1$ and $\vec{r}_2$ respectively is

$$dU_C = (\rho_1 d^3x_1)(\rho_2 d^3x_2) V_C(\vec{r}_{12}). \tag{2.4}$$

The average electrostatic energy U_C is then obtained by integrating Eq. (2.4) over the entire sphere:

$$\begin{aligned} U_C &= \int \rho_1 d^3x_1 \int \rho_2 d^3x_2 \, V_C(\vec{r}_{12}) \\ &= e_0^2 \rho^2 \int_0^R dr_2 \, r_2^2 \int_0^R dr_1 \, r_1^2 \int d\Omega_1 \int d\Omega_2 \left(\frac{1}{r_{12}}\right) \\ &= \frac{6}{5} \frac{e_0^2}{R}, \end{aligned} \tag{2.5}$$

where $r_1 = |\vec{r}_1|$ and $r_2 = |\vec{r}_2|$ are measured from the center of the sphere, and $d\Omega = d\cos\theta d\varphi$. The result in Eq. (2.5) represents the average interaction energy of a single pair of charges having a uniform probability distribution in a spherical volume of radius R.

Although the 6-dimensional integral arising from the 2-body potential energy in Eq. (2.2) is straightforward, its generalization to the many-body potentials that arise from neutrino-exchange becomes increasingly cumbersome. We will return below to discuss an alternative method for evaluating the integral over a sphere, which generalizes more naturally to the many-body case.

C. Combinatorics: W_C

As noted above, the result in Eq. (2.5) gives the energy U_C for a single pair of charges. For a nucleus containing Z charges the number of pairs that can be formed is $Z(Z-1)/2$, so that the final expression for the total energy W_C is

$$W_C = \frac{1}{2}Z(Z-1)U_C = \frac{3}{5}Z(Z-1)\frac{e_0^2}{R}, \tag{2.6}$$

in agreement with Eq. (2.1). For application to the k-body problem ($k = 2, 3, 4, \ldots$) the combinatoric factor $Z(Z-1)/2$ generalizes to the binomial coefficient [21] defined in Eq. (1.4) which counts the number of combinations of k objects that can be formed from N objects. Evidently

$$\binom{Z}{2} = \frac{Z(Z-1)}{2}, \tag{2.7}$$

which reproduces the result in Eq. (2.6).

As we noted in the Introduction, the fact that the net k-body contribution is proportional to the binomial coefficient $\binom{N}{k}$ was pointed out by Primakoff and Holstein in their classic paper [21] on many-body electromagnetic forces. Subsequently Chanmugam and Schweber [22] re-examined the question of many-body forces in electromagnetism. They observed that since the binomial coefficient for small k grows as

$$\binom{N}{k} \simeq \frac{N^k}{k!}, \tag{2.8}$$

this factor may make many-body ($k > 2$) electromagnetic effects important in white dwarfs, where the electron density can be high. For neutrino-exchange amplitudes the presence of the binomial coefficient in the final expression for the interaction energy is what eventually leads to the large neutrino-exchange energy-density referred to in the Introduction. Specifically, each neutron in a neutron star carries a non-zero weak charge which couples the neutron to the $\nu\bar{\nu}$ current, so that for a neutron star N is of order 10^{57}. By way of comparison, the electric charge Z of a neutron star is at most of order 10^{36} [26], as noted in the Introduction. It follows that the enhancement of the many-body amplitudes arising from the binomial

coefficient is a much more significant effect for neutrino exchange than it would be for the corresponding electrostatic case.

In the ensuing discussion it will be helpful to recall that the binomial coefficient $\binom{N}{k}$ is a decreasing function of increasing k for fixed N: From Eq. (1.4),

$$\frac{\binom{N}{k+1}}{\binom{N}{k}} = \frac{N-k}{k+1}. \tag{2.9}$$

Hence the k-body interaction is proportional to a combinatoric factor which, although large, is decreasing monotonically as k increases. From the discussion in Appendix B we see that the weak interaction energy arising from neutrino-exchange can be expressed as a sum over k-body contributions $W^{(k)}$ as in Eq. (B1), each of which will be proportional to $\binom{N}{k}$. The fact that the ratio in Eq. (2.9) is less than unity when $k > (N-1)/2$ helps to explain why $\sum_k W^{(k)}$ is dominated by a few terms with $k \simeq N$, as we discuss in Sec. V below.

D. Alternate Method of Integration

Having outlined the steps that lead to Eq. (2.1), we return to discuss an alternate method for evaluating the integral of the Coulomb potential in Eq. (2.2) over a spherical volume. Let $\mathcal{P}_3(r)$ denote the normalized probability density for finding two points randomly chosen in a uniform 3-dimensional sphere to be a distance $r \equiv r_{12}$ apart. As discussed in Appendix C, the average value $\langle g \rangle$ of any function $g(r)$ taken over a 3-dimensional spherical volume is then given by

$$\langle g \rangle = \int_0^{2R} dr \mathcal{P}_3(r) g(r), \tag{2.10}$$

where

$$\int_0^{2R} dr \mathcal{P}_3(r) = 1. \tag{2.11}$$

The functional form of $\mathcal{P}_3(r)$, and its generalization $\mathcal{P}_n(r)$ for an n-dimensional ball of radius R, are discussed in Appendix C. We henceforth drop the subscript 3 when working in 3-dimensions ($\mathcal{P} \equiv \mathcal{P}_3$). From Eq. (C7), with $r = 2Rs$,

$$\begin{aligned}\mathcal{P}(r) &= \frac{3r^2}{R^3} - \frac{9}{4}\frac{r^3}{R^4} + \frac{3}{16}\frac{r^5}{R^6} \\ &= \frac{3r^2}{R^3}\left[1 - \frac{3}{2}\left(\frac{r}{2R}\right) + \frac{1}{2}\left(\frac{r}{2R}\right)^3\right].\end{aligned} \tag{2.12}$$

Since $2R \geq r \geq 0$, it is sometimes convenient to introduce the scaled dimensionless variable s defined above which satisfies $1 \geq s \geq 0$. Plots of $\mathcal{P}(s)$ and its derivative $\mathcal{P}'(s)$ are given in Fig. 3. Eq. (2.10) can be viewed as an *analytic* Monte Carlo calculation of $\langle g \rangle$.

Returning to the Coulomb problem we wish to calculate $\langle e_0^2/r \rangle$ by this method. From Eq. (2.10),

$$e_0^2\langle 1/r \rangle = e_0^2 \int_0^{2R} dr \left(\frac{3r^2}{R^3} - \frac{9r^3}{4R^4} + \frac{3r^5}{16R^6}\right)\frac{1}{r} = \frac{6}{5}\frac{e_0^2}{R}, \tag{2.13}$$

in agreement with Eq. (2.5). The advantage of this approach is that, where applicable, it replaces the 6-dimensional integral in Eq. (2.5) by the 1-dimensional integral in Eq. (2.13). Other useful results are given in Appendix C.

E. Application to Massive Electrodynamics

We conclude the previous discussion by using Eq. (2.10) to calculate the electrostatic energy of a spherical charge distribution in massive electrodynamics, i.e., for the case of a photon with a non-zero mass μ. This calculation completes the analogy between the electrostatic energy arising from photon exchange, and the weak energy arising from neutrino exchange when $m \neq 0$. When $\mu \neq 0$ the Coulomb potential in Eq. (2.2) is replaced by the Yukawa potential [31]

$$V_Y(r_{12}) = \frac{e_0^2}{r_{12}} e^{-\mu r_{12}}, \tag{2.14}$$

and the result in Eq. (2.13) generalizes to

$$e_0^2\langle e^{-\mu r}/r \rangle = \frac{6}{5}\frac{e_0^2}{R} F(\mu R), \tag{2.15}$$

where

$$F(\mu R) = \frac{15}{4}\left[\frac{2}{3(\mu R)^2} - \frac{1}{(\mu R)^3} + \frac{1}{(\mu R)^5}\right] - \frac{15}{4}e^{-2\mu R}\left[\frac{1}{(\mu R)^3} + \frac{2}{(\mu R)^4} + \frac{1}{(\mu R)^5}\right]. \quad (2.16)$$

The "form factor" $F(\mu R)$ incorporates all the modifications arising from $\mu \neq 0$, and has the property that $F(0) = 1$.

We note in passing that the result in Eqs. (2.15) and (2.16) is also of interest in connection with recent work setting limits on new forces co-existing with electromagnetism [31]. Although limits on the photon mass are quite stringent, the corresponding limits on a new vector force co-existing with electromagnetism are less restrictive. If e_0 in Eq. (2.15) were replaced by the corresponding unit of charge f for the new field, then Eq. (2.15) could be used to set limits on f^2 and μ in appropriate systems, in a manner similar to that described below for the neutrino mass m.

F. Quantum Mechanical Effects

The preceding calculation also serves to clarify another issue which arises in neutrino-exchange, namely, why the charge distribution can be treated classically even in an object such as a nucleus or a neutron star where quantum effects are important. In particular one may ask what role the Pauli exclusion principle plays for the protons in a nucleus or the neutrons in a neutron star in the presence of a long-range force. For present purposes we can invoke an argument due to Fermi [32] to demonstrate that the self-energy of a nucleus or neutron star arising from long-range forces can in fact be approximated by the classical result, as we have done. (We will return in Sec. V below to discuss the effects of the Pauli principle for the neutrinos.)

III. THE MANY-BODY POTENTIALS ARISING FROM NEUTRINO-EXCHANGE

Following the discussion in Sec. II, we present in this section a derivation of the many-body spin-independent potentials arising from neutrino-exchange. We will begin by using

the Hartle formalism [18,29] to re-derive the original 2-body result of Feinberg and Sucher (FS) arising from the diagram in Fig. 1 [15–17]. It will then be shown that k-body potentials where k is odd make no contribution to the neutrino-exchange energy of a spherical neutron star. After reproducing the 4-body result of Hartle [18], we derive the 6-body potential which is then used to infer the relevant combinatoric factors in the general k-body potential. For the sake of definiteness we assume that the external fermions are neutrons, as would be the case in a neutron star, and we therefore suppress the corresponding subscripts on V and W.

Our starting point is the Schwinger formula in Eq. (B34),

$$W = \frac{i}{2\pi}\mathrm{Tr}\left\{\int_{-\infty}^{\infty} dE \ \ln[1 + \frac{G_F a_n}{\sqrt{2}} N_\mu \gamma_\mu (1+\gamma_5) S_F^{(0)}(E)]\right\}, \tag{3.1}$$

where Tr denotes the generalized trace defined by Eq. (B13), and $\ln[1+\cdots]$ represents the infinite series

$$\ln(1+\Delta) = -\sum_{k=1}^{\infty} (-1)^k \frac{\Delta^k}{k}. \tag{3.2}$$

The factor $1/k$ can be understood as the product of $1/k!$ arising from the perturbation expansion of $\exp(i\mathcal{L}_I(x))$ where $\mathcal{L}_I(x)$ is given in Eq. (B16), and $(k-1)!$ which represents the number of ways that the k external currents can be attached to the closed neutrino loop. There is an additional factor of $k!$ which counts the number of ways that the momenta carried by the external currents can be assigned to the k vertices. The net result is that the expansion of $\ln[1+\cdots]$ in the Schwinger formula gives rise to $(k-1)!$ topologically distinct diagrams, but there is no overall k-dependent numerical coefficient [33]. For $k=4$ there are 6 independent diagrams: These are given by the 3 shown in Fig. 2 along with an additional 3 diagrams obtained from those shown by reversing the direction of the internal neutrino momentum, as in Fig. 4. For $k=2$, $(k-1)! = 1$ which agrees with our expectation that there be only one diagram in order G_F^2, since both senses of the neutrino momentum are topologically equivalent.

A. The 2-Body Potential

Combining Eqs. (3.1) and (3.2) the $k=2$ contribution to W is given by

$$W^{(2)} = \frac{-i}{2\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^2 \int d^3x_1 d^3x_2 \int_{-\infty}^{\infty} dE$$
$$\times \mathrm{tr}\left\{\gamma_\mu(1+\gamma_5)S_F^{(0)}(\vec{r}_{12},E)\gamma_\nu(1+\gamma_5)S_F^{(0)}(\vec{r}_{21},E)\right\} N_\mu(x_1)N_\nu(x_2), \quad (3.3)$$

where $\vec{r}_{12} = (\vec{x}_1 - \vec{x}_2)$, $N_\mu(x_1)$ and $N_\nu(x_2)$ denote the external neutron currents at x_1 and x_2, and $S_F^{(0)}(\vec{r}_{ij},E)$ is given by Eq. (B42). The overall minus sign arises from the expansion in Eq. (3.2), and tr denotes the trace over the Dirac matrices in $\{\cdots\}$. The factors of $(1+\gamma_5)$ can be anticommutated past $S_F^{(0)}$ and, using the relation

$$(1+\gamma_5)^2 = 2(1+\gamma_5), \quad (3.4)$$

we have

$$\mathrm{tr}\{\cdots\} = 2\mathrm{tr}\left\{S_F^{(0)}(\vec{r}_{21},E)\gamma_\mu S_F^{(0)}(\vec{r}_{12},E)\gamma_\nu(1+\gamma_5)\right\}. \quad (3.5)$$

Since we are interested in computing the self energy of a static collection of neutrons (which are assumed to have no net polarization), the neutron currents are given by

$$N_\mu(x_1) = i\rho_1\delta_{\mu 4}; \qquad N_\nu(x_2) = i\rho_2\delta_{\nu 4}, \quad (3.6)$$

where $\rho_1 = \rho_2 = \rho$ is the number density of neutrons in the neutron star, and the factors of i arise from Eq. (A4). Combining Eqs. (3.5), (3.6), and (A8) we see that the term containing γ_5, whose trace is proportional to $\epsilon_{\mu\nu\lambda\rho}$, makes no contribution. The remaining terms give

$$W^{(2)} = \frac{-(i)^3}{\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^2 \int \rho_1 d^3x_1 \int \rho_2 d^3x_2 \int_{-\infty}^{\infty} dE$$
$$\times\ \mathrm{tr}\left[\gamma_\alpha\gamma_4\gamma_\beta\gamma_4\right]\eta_\alpha(21)\Delta_F(\vec{r}_{21},E)\eta_\beta(12)\Delta_F(\vec{r}_{12},E). \quad (3.7)$$

Using

$$\mathrm{tr}[\gamma_\alpha\gamma_4\gamma_\beta\gamma_4] = 4(2\delta_{\alpha 4}\delta_{\beta 4} - \delta_{\alpha\beta}), \quad (3.8)$$

the expression in curly brackets in Eq. (3.5) can be written as

$$\begin{aligned}\{(3.5)\} &= 4[2\eta_4(21)\eta_4(12) - \eta(21)\cdot\eta(12)] \\ &= 4[E^2 - \vec{\partial}_{12}\cdot\vec{\partial}_{21}],\end{aligned} \tag{3.9}$$

where $\vec{\partial}_{12} \equiv \partial/\partial\vec{r}_{12}$. Combining Eqs. (3.9) and (3.7) we have

$$\begin{aligned}W^{(2)} = &\frac{4i}{\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^2 \int \rho_1 d^3x_1 \int \rho_2 d^3x_2 \int_{-\infty}^{\infty} dE \\ &\times \left\{E^2 \Delta_F(\vec{r}_{21},E)\Delta_F(\vec{r}_{12},E) - \vec{\partial}_{12}\cdot\vec{\partial}_{21}\Delta_F(\vec{r}_{21},E)\Delta_F(\vec{r}_{12},E)\right\} \\ = &\frac{4i}{\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^2 \int \rho_1 d^3x_1 \int \rho_2 d^3x_2 \int_{-\infty}^{\infty} dE \\ &\times \left(\frac{i}{4\pi}\right)^2 \left\{E^2 \frac{e^{i|E|(r_{12}+r_{21}+i\epsilon)}}{r_{12}r_{21}} - \vec{\partial}_{12}\cdot\vec{\partial}_{21}\left(\frac{e^{i|E|(r_{12}+r_{21}+i\epsilon)}}{r_{12}r_{21}}\right)\right\}.\end{aligned} \tag{3.10}$$

We note from Eq. (B42) that the operators $\vec{\partial}_{12}$ and $\vec{\partial}_{21}$ act on the respective coordinates $\vec{r}_{12}$ and $\vec{r}_{21}$ as if these were independent, notwithstanding the fact that $\vec{r}_{12} + \vec{r}_{21} = 0$. This applies as well to all the derivative terms that appear in the k-body amplitudes.

Following Hartle [29] the integral over E can be evaluated by considering the functions $\bar{I}_n(z)$ defined by

$$\begin{aligned}\bar{I}_n(z) &= \int_{-\infty}^{\infty} dE\ E^n e^{i|E|(z+i\epsilon)} \\ &= \begin{cases} 2\int_0^\infty dE\ E^n\ e^{i|E|(z+i\epsilon)} \equiv I_n(z) & \text{even } n \\ 0 & \text{odd } n, \end{cases}\end{aligned} \tag{3.11}$$

where $z = r_{12} + r_{21}$. Since $|E|$ is an even function of E, $\bar{I}_n(z)$ is nonzero only for even values of n. An elementary integration gives

$$I_0(z) = \frac{2i}{z+i\epsilon}, \tag{3.12}$$

and differentiating Eqs. (3.11) and (3.12) with respect to z leads to

$$-i\frac{dI_0(z)}{dz} = I_1(z) = \frac{-2}{(z+i\epsilon)^2}. \tag{3.13}$$

Continuing in this way we find [29]

$$I_n(z) = \frac{2i^{n+1}n!}{(z+i\epsilon)^{n+1}}. \tag{3.14}$$

Combining Eqs. (3.10) and (3.14) allows $W^{(2)}$ to be written as

$$
\begin{aligned}
W^{(2)} = \int \rho_1 d^3x_1 \int \rho_2 d^3x_2 \Bigg\{ \left(\frac{-1}{\pi^3}\right) \left(\frac{G_F a_n}{\sqrt{2}}\right)^2 \\
\times \left[\frac{1}{r_{12} r_{21} (r_{12}+r_{21})^3} + \frac{1}{2} \vec{\partial}_{12} \cdot \vec{\partial}_{21} \frac{1}{r_{12} r_{21} (r_{12}+r_{21})} \right] \Bigg\}.
\end{aligned}
\tag{3.15}
$$

We note from Eq. (2.5) that the integrand in Eq. (3.15) has the same form as in the analogous electromagnetic case, with the quantity in curly brackets in Eq. (3.15) corresponding to the 2-body potential $V^{(2)}(r_{12})$. Thus

$$
V^{(2)}(r_{12}) = \frac{-1}{\pi^3} \left(\frac{G_F a_n}{\sqrt{2}}\right)^2 \left[\frac{1}{r_{12} r_{21} (r_{12}+r_{21})^3} + \frac{1}{2} \vec{\partial}_{12} \cdot \vec{\partial}_{21} \frac{1}{r_{12} r_{21} (r_{12}+r_{21})} \right]. \tag{3.16}
$$

Since $r \equiv r_{12} = r_{21}$, first term in square brackets in Eq. (3.16) reduces to $1/8r^5$. In the second term we note that the gradients act on a function which depends only on r_{12} and r_{21}, and hence we can write

$$
\vec{\partial}_{12} \cdot \vec{\partial}_{21} = \left(\hat{r}_{12} \frac{\partial}{\partial r_{12}} \right) \cdot \left(\hat{r}_{21} \frac{\partial}{\partial r_{21}} \right) = \hat{r}_{12} \cdot \hat{r}_{21} \frac{\partial}{\partial r_{12}} \frac{\partial}{\partial r_{21}} = -\frac{\partial}{\partial r_{12}} \frac{\partial}{\partial r_{21}}. \tag{3.17}
$$

Using Eq. (3.17) the expression in square brackets in Eq. (3.16) can be written as

$$
[(3.16)] = \frac{1}{8r^5} - \frac{5}{8r^5} = \frac{-1}{2r^5}, \tag{3.18}
$$

and hence,

$$
V^{(2)}(r) = +\frac{G_F^2 a_n^2}{4\pi^3} \frac{1}{r^5}. \tag{3.19}
$$

Eq. (3.19) gives the original FS result [15–17] when we set $a_n = 1$, which is the value appropriate to the charged-current model of the weak interaction assumed by FS.

For later purposes it is interesting to note that the functional form of $V^{(2)}(r)$ can be inferred on dimensional grounds, as noted originally by Feinberg and Sucher [15]. The only dimensional quantities upon which a static neutrino-exchange potential can depend are G_F, r and (possibly) the masses of the external particles. However, in the non-relativistic limit appropriate to a static potential, bilinear covariants such as $\bar{u}(p')\gamma_\lambda(1+\gamma_5)u(p)$ are independent of the mass of the fermion characterized by the spinor $u(p)$. Thus the only

relevant dimensional parameters are G_F and r and, since the 2-body operator is proportional to G_F^2, it follows that $V^{(2)}(r) \propto G_F^2/r^5$. Implicit in this argument is the assumption that no other dimensional parameters are present and, since the standard model is renormalizable, this will indeed be the case. This argument holds even in the framework of the (non-renormalizable) charged-current model originally assumed by FS, since the regularization procedure employed by FS to extract the long-distance behavior of $V^{(2)}(r)$ introduces no additional mass parameters.

We conclude the discussion of the 2-body potential by demonstrating quantitatively that its effects are too small to be detected at present in any known system. Let us consider the analog of Eq. (3.19) for electrons, for which the "background" gravitational interaction would be smallest. Reinstating $\hbar$ and c, and substituting $a_n \to a_e = (2\sin^2\theta_W + 1/2)$, where [6]

$$\sin^2\theta_W = 0.2319(5), \tag{3.20}$$

we find

$$V^{(2)}(r) = \frac{(2\sin^2\theta_W + 1/2)^2}{4\pi^3}\frac{G_F^2}{\hbar c}\frac{1}{r^5}. \tag{3.21}$$

Using [6]

$$\frac{G_F}{(\hbar c)^3} = 1.16639(2) \times 10^{-5}\mathrm{GeV}^{-2} \tag{3.22}$$

leads to

$$V^{(2)}(r) = 3 \times 10^{-82}\frac{\mathrm{eV}}{(r/1\ \mathrm{m})^5}. \tag{3.23}$$

The magnitude of the corresponding force $\vec{F}_{12}(r) = -\vec{\nabla}V^{(2)}(r)$ is then given by

$$|\vec{F}_{12}(r)| = \frac{2 \times 10^{-100}}{(r/1\ \mathrm{m})^6}\ \mathrm{Newtons}. \tag{3.24}$$

To appreciate how weak this force is, we compare $|\vec{F}_{12}|$ to the gravitational force $\vec{F}_{12}^{grav}$ at a nominal separation $r = 1$ km,

$$\frac{|\vec{F}_{12}(r = 1\text{ km})|}{|\vec{F}_{12}^{grav}(r = 1\text{ km})|} = 4 \times 10^{-42}. \tag{3.25}$$

As r decreases $|\vec{F}_{12}(r)|$ increases more rapidly than does $|\vec{F}_{12}^{grav}|$, and these forces become equal at $r = 5 \times 10^{-8}$m. However, at this separation electromagnetic and weak forces arising from Z^0-exchange would be much larger than $|\vec{F}_{12}|$, and hence there appears to be no distance scale over which the presence of the 2-body neutrino-exchange interaction can be directly detected [15,23].

B. The k-Body Contribution when k is Odd

In this subsection we show that there is no 3-body static potential arising from neutrino-exchange. This result is then generalized by demonstrating that for odd values of $k \geq 5$, the k-body potential exists but averages to zero when integrated over a spherical volume. The net result is that the neutrino-exchange energy of a spherical nucleus or neutron star is given by a sum of k-body contributions where k is even.

As noted previously, there are two independent diagrams for $k = 3$ and these are shown in Fig. 5. Expanding Eq. (3.1) to $\mathcal{O}(G_F^3)$ we find for the contribution from diagram 5(a)

$$\begin{aligned} W_a^{(3)} = \frac{i}{2\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^3 &\int d^3x_1 d^3x_2 d^3x_3 \int_{-\infty}^{\infty} dE \\ &\times \Big[\mathrm{tr}\{\gamma_\lambda(1+\gamma_5)S_F^{(0)}(12)\gamma_\mu(1+\gamma_5)S_F^{(0)}(23)\gamma_\nu(1+\gamma_5)S_F^{(0)}(31)\} \\ &\times N_\lambda(x_1)N_\mu(x_2)N_\nu(x_3)\Big], \end{aligned} \tag{3.26}$$

where $S_F^{(0)}(12) \equiv S_F^{(0)}(\vec{r}_{12}, E)$ etc. Using Eq. (3.4) the expression in $\{\cdots\}$ for diagram 5(a) can be simplified to

$$\mathrm{tr}\{\text{diagram } 5(a)\} = 4\ \mathrm{tr}\{\gamma_\lambda S_F^{(0)}(12)\gamma_\mu S_F^{(0)}(23)\gamma_\nu S_F^{(0)}(31)(1-\gamma_5)\}. \tag{3.27}$$

The contribution $W_b^{(3)}$ from diagram 5(b) is given by the same expression as in Eq. (3.26) except that $\mathrm{tr}\{\cdots\}$ is replaced by

$$\mathrm{tr}\{\text{diagram } 5(b)\} = 4\ \mathrm{tr}\{\gamma_\lambda S_F^{(0)}(13)\gamma_\nu S_F^{(0)}(32)\gamma_\mu S_F^{(0)}(21)(1-\gamma_5)\}. \tag{3.28}$$

The familiar arguments used to derive Furry's theorem [34] can now be adapted to show that the 3-body contribution vanishes. We introduce the charge-conjugation matrix C defined by

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T, \tag{3.29a}$$

$$C^{-1}(1-\gamma_5)C = (1-\gamma_5)^T, \tag{3.29b}$$

where the superscript T denotes the transpose matrix. In the Dirac-Pauli conventions C is given (up to an overall phase) by $C = \gamma_2\gamma_4$. By virtue of Eqs. (3.29a) and (3.29b), the neutrino propagator in Eq. (B42) satisfies

$$C^{-1}S_F^{(0)}(\vec{r}_{ij}, E)C = -\gamma^T \cdot \eta(ij)\Delta_F(\vec{r}_{ij}, E) = S_F^{(0)T}(-\vec{r}_{ij}, -E) \equiv S_F^{(0)T}(-ij). \tag{3.30}$$

Using Eqs. (3.29a), (3.29b), and (3.30), Eq. (3.28) can be rewritten in the form

$$\begin{aligned}
&4\ \mathrm{tr}\{\gamma_\lambda S_F^{(0)}(13)\gamma_\nu S_F^{(0)}(32)\gamma_\mu S_F^{(0)}(21)(1-\gamma_5)\} \\
&\qquad = 4\mathrm{tr}\{(-\gamma_\lambda^T)S_F^{(0)T}(-13)(-\gamma_\nu^T)S_F^{(0)T}(-32)(-\gamma_\mu^T)S_F^{(0)T}(-21)(1-\gamma_5)^T\} \\
&\qquad = -4\ \mathrm{tr}\{\gamma_\lambda S_F^{(0)}(12)\gamma_\mu S_F^{(0)}(23)\gamma_\nu S_F^{(0)}(31)(1+\gamma_5)\}.
\end{aligned} \tag{3.31}$$

The last step of Eq. (3.31) follows by noting from Eq. (3.30) that we can replace $-\vec{r}_{ij}$ by $\vec{r}_{ji}$ in $S_F^{(0)}(-ij)$, so that $S_F^{(0)}(-ij)$ is the same as $S_F^{(0)}(ji)$ except for the sign of E. However, by virtue of the symmetric limits of the integration over E, only even powers of E contribute when the products of the neutrino propagators are expanded. Since these give the same contributions for $\pm E$, we can effectively replace $S_F^{(0)}(-ij)$ by $S_F^{(0)}(ji)$ in Eq. (3.31). Combining Eqs. (3.27) and (3.31) we see that the terms independent of γ_5 cancel exactly, which is Furry's theorem, while the terms proportional to γ_5 add yielding

$$\begin{aligned}
W^{(3)} &= W_a^{(3)} + W_b^{(3)} \\
&= \frac{-i}{2\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^3 \int d^3x_1 d^3x_2 d^3x_3 \int_{-\infty}^{\infty} dE \\
&\quad \times \left[8\ \mathrm{tr}\{\gamma_\lambda S_F^{(0)}(12)\gamma_\mu S_F^{(0)}(23)\gamma_\nu S_F^{(0)}(31)\gamma_5\}N_\lambda(x_1)N_\mu(x_2)N_\nu(x_3)\right].
\end{aligned} \tag{3.32}$$

Using Eq.(A8) we see that the trace in Eq. (3.32) is proportional to the Levi-Civita tensor $\epsilon_{\mu\nu\lambda\rho}$, which reduces in the non-relativistic limit to the permutation symbol ϵ_{ijl}. It then follows that the 3-body potential $V^{(3)}(\vec{r}_{12}, \vec{r}_{13}, \vec{r}_{23})$ will contain terms of the form

$$\epsilon_{ijl}(\vec{r}_{12})_i(\vec{r}_{31})_j(\vec{r}_{23})_l = \vec{r}_{12} \cdot (\vec{r}_{31} \times \vec{r}_{23}) = 0, \tag{3.33}$$

since $\vec{r}_{12} + \vec{r}_{31} + \vec{r}_{23} = 0$. Thus there is no 3-body potential arising from neutrino exchange.

The preceding arguments can be generalized to show that one can ignore the contributions from k-body potentials when $k \geq 5$ is odd. As we discuss in the following subsection, the k-body potential arising from the expansion of Eq. (3.1) receives contributions from $\frac{1}{2}(k-1)!$ topologically distinct pairs of diagrams, each being sum of two diagrams, representing the two senses of the neutrino-loop momentum. For the j-th such diagram the sum of the contributions from the two senses of the neutrino-loop momentum gives rise to an analog of Eq. (3.32) which, if k is odd, has the form

$$\begin{aligned} W_j^{(k)} = {} & \frac{-i}{2\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^k \int d^3x_1 d^3x_2 \cdots d^3x_k \int_{-\infty}^{\infty} dE \\ & \times 2^k \mathrm{tr}\{\gamma_\lambda S_F^{(0)}(12)\gamma_\mu S_F^{(0)}(23) \cdots S_F^{(0)}(k1)\gamma_5\} N_\lambda(x_1) \cdots N_\mu(x_k). \end{aligned} \tag{3.34}$$

Evaluation of $\mathrm{tr}\{\cdots\}$ leads again to an expression which is proportional to $\epsilon_{\mu\nu\lambda\rho}$, and hence to ϵ_{ijl} in the nonrelativistic limit. For $k \geq 5$ an expression such as Eq. (3.33) need not vanish, since ϵ_{ijl} can be contracted with 3 linearly independent vectors, and hence the corresponding potential is in general nonzero. However, the integral of any such potential over a spherical volume *is* zero: In analogy to the electrostatic case discussed in Sec. II, the integrated energy can depend only on G_F and the radius R of the sphere. Since a nonzero scalar product cannot be formed utilizing only ϵ_{ijl}, R, and $\vec{R} = \hat{R}R$, the k-body contribution to the energy of a spherical charge distribution must vanish when k is odd. This conclusion does not necessarily hold for an asymmetric charge distribution, but in the present context any contributions from k-odd terms would be proportional to the (presumably small) deviations of the matter distribution in a white dwarf or neutron star from spherical symmetry.

C. The 4-Body Potential

Following the discussion in Appendix B we note that the expansion of Eq. (B34) in powers of G_F leads in order G_F^k to the one-loop diagrams shown in Fig. 4, each having k vertices. Since there are $(k-1)!/2$ distinct ways that the integers $1, 2, 3, \ldots, k$ can be arranged on the perimeter of a circle, there are altogether $(k-1)!$ topologically distinct diagrams in order k, where the additional factor of 2 takes account of the two senses of the neutrino-loop momentum. In each order we will refer to the diagram in which the vertices are labeled sequentially $1, 2, 3, \ldots, k$ as the"standard" diagram. When computing the k-body potential, each of the $(k-1)!/2$ possible sequences of integers leads to a distinct potential arising from the sum of the two diagrams with opposite senses of the neutrino-loop momentum. Consider, for example, the three topologically distinct diagrams corresponding to $k = 4$ shown in Fig. 2. Since the dependence of any term in the potential on $\vec{r}_{ij}$ arises from the neutrino propagator connecting i and j, the standard diagram, Fig. 2(a), will be a function of only the variables $\vec{r}_{12}, \vec{r}_{23}, \vec{r}_{34}$, and $\vec{r}_{41}$, but not of $\vec{r}_{24}$ or $\vec{r}_{13}$, which are not connected by neutrino propagators. Thus the 3 pairs of diagrams represented in Fig. 4 give rise to 3 distinct contributions to the total potential in order G_F^4, as we show explicitly below. However, on symmetry grounds each of these 3 contributions leads to the same result when integrated over a spherical volume. Hence in practice it suffices to evaluate the pair of diagrams containing the standard sequence of vertices $1, 2, 3, \ldots, k$, in the anticipation that the integrated result will eventually be multiplied by $(k-1)!/2$ to obtain the total contribution in order k. Since any specific set of k particles can be chosen in $\binom{N}{k}$ ways from among N particles, there is an additional factor of $\binom{N}{k}$ present in the final result, and it is this factor which is ultimately responsible for the large neutrino-exchange energy-density.

We can summarize the preceding discussion and Appendix B as follows: The expansion of the Schwinger formula in Eq. (B34) leads to a set of irreducible 1-loop Feynman diagrams which describe the $\mathcal{O}(G_F^k)$ contribution to the k-body potential. The k-body potential can also receive contributions from diagrams containing higher powers of G_F, but these are

suppressed both by G_F and by various mass factors. Hence for practical purposes we need consider only the one-loop diagrams arising from the expansion of Eq. (B34). Each of these contains k vertices, which represent in configuration space the coordinates $\vec{r}_i (i = 1, 2, \ldots, k)$ of the k particles. The k vertices are connected by k legs representing the variables $r_{ij} = |\vec{r}_{ij}| = |\vec{r}_i - \vec{r}_j|$. There are $k(k-1)/2$ such variables that arise, and $(k-1)!/2$ topologically distinct diagrams, although each diagram depends on a subset containing only k of the variables. Notwithstanding the fact that the variables $\vec{r}_{ij}$ satisfy a constraint of the form

$$\vec{r}_{12} + \vec{r}_{23} + \vec{r}_{34} + \cdots + \vec{r}_{k1} = 0, \tag{3.35}$$

the evaluation of the k-body potential for $k > 2$ should be carried out treating the $\vec{r}_{ij}$ as if they were in fact independent.

We turn next to the detailed form of the 4-body contribution. It follows from the preceding discussion that we can confine our attention to the standard diagrams, Figs. 4(a) and 4(a′), which incorporate both senses of the neutrino loop momentum. Expanding Eq. (B34) to $\mathcal{O}(G_F^4)$ and using Eq. (3.4) we find for the contribution from diagram 4(a),

$$\begin{aligned} W_a^{(4)} = \frac{-i}{2\pi} \left(\frac{G_F a_n}{\sqrt{2}} \right)^4 \int d^3x_1 d^3x_2 d^3x_3 d^3x_4 \int_{-\infty}^{\infty} dE \\ \times 2^3 \mathrm{tr} \left\{ \gamma_\mu S_F^{(0)}(14) \gamma_\sigma S_F^{(0)}(43) \gamma_\lambda S_F^{(0)}(32) \gamma_\nu S_F^{(0)}(21)(1-\gamma_5) \right\} T_{\mu\nu\lambda\sigma}(x_1, x_2, x_3, x_4), \end{aligned} \tag{3.36}$$

where

$$T_{\mu\nu\lambda\sigma}(x_1, x_2, x_3, x_4) \equiv N_\mu(x_1) N_\nu(x_2) N_\lambda(x_3) N_\sigma(x_4). \tag{3.37}$$

By use of Eqs. (3.29a) and (3.29b), $\mathrm{tr}\{\cdots\}$ in Eq. (3.36) can be written in the form

$$\mathrm{tr}\{\text{diagram 4(a)}\} = \mathrm{tr}\{S_F^{(0)}(12)\gamma_\nu S_F^{(0)}(23)\gamma_\lambda S_F^{(0)}(34)\gamma_\sigma S_F^{(0)}(41)\gamma_\mu(1-\gamma_5)\}. \tag{3.38}$$

The contribution from diagram 4(a′), corresponding to the standard diagram with the loop momentum reversed, has the same form as Eq. (3.36) except that $\mathrm{tr}\{\cdots\}$ is replaced by

$$\mathrm{tr}\{\text{diagram 4(a}')\} = \mathrm{tr}\{S_F^{(0)}(12)\gamma_\nu S_F^{(0)}(23)\gamma_\lambda S_F^{(0)}(34)\gamma_\sigma S_F^{(0)}(41)\gamma_\mu(1+\gamma_5)\}. \tag{3.39}$$

We see that when the contributions in Eqs. (3.38) and (3.39) are added the terms proportional to γ_5 now cancel, whereas they added in the 3-body case. Similarly, the terms independent of γ_5, which cancelled previously to yield Furry's theorem, now add to give

$$\begin{aligned} W^{(4)} &= W_a^{(4)} + W_{a'}^{(4)} \\ &= \frac{-i}{2\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^4 2^4 \int d^3x_1 ... d^3x_4 \int_{-\infty}^{\infty} dE \\ &\quad \times \mathrm{tr}\{S_F^{(0)}(12)\gamma_\nu S_F^{(0)}(23)\gamma_\lambda S_F^{(0)}(34)\gamma_\sigma S_F^{(0)}(41)\gamma_\mu\} T_{\mu\nu\lambda\sigma}(x_1, \ldots, x_4). \end{aligned} \tag{3.40}$$

As in the 2-body case we are interested in the static potential which arises from

$$T_{\mu\nu\lambda\sigma}(x_1, \ldots, x_4) = (i)^4 \rho_1\rho_2\rho_3\rho_4 \delta_{\mu 4}\delta_{\nu 4}\delta_{\lambda 4}\delta_{\sigma 4}, \tag{3.41}$$

and hence the trace in Eq. (3.40) is proportional to

$$\mathrm{tr}\{(3.40)\} \sim \mathrm{tr}[\gamma \cdot \eta(12)\gamma_4 \gamma \cdot \eta(23)\gamma_4 \gamma \cdot \eta(34)\gamma_4 \gamma \cdot \eta(41)\gamma_4], \tag{3.42}$$

where our notation is the same as in Eq. (3.7). It is convenient to simplify the trace by writing

$$\gamma_4 \gamma \cdot \eta(12) = -(\vec{\gamma} \cdot \vec{\partial}_{12} + \gamma_4 E)\gamma_4 \equiv -\gamma \cdot \bar{\eta}(12)\gamma_4, \tag{3.43}$$

so that $\mathrm{tr}[\cdots]$ in Eq. (3.42) assumes the form

$$\begin{aligned} \mathrm{tr}[\cdots] &= \mathrm{tr}[\gamma \cdot \bar{\eta}(12)\gamma \cdot \eta(23)\gamma \cdot \bar{\eta}(34)\gamma \cdot \eta(41)] \\ &= 4\{E^4 - E^2(\vec{\partial}_{12} \cdot \vec{\partial}_{23} + \vec{\partial}_{23} \cdot \vec{\partial}_{34} + \vec{\partial}_{34} \cdot \vec{\partial}_{41} + \vec{\partial}_{12} \cdot \vec{\partial}_{34} + \vec{\partial}_{12} \cdot \vec{\partial}_{41} + \vec{\partial}_{23} \cdot \vec{\partial}_{41}) \\ &\quad + [(\vec{\partial}_{12} \cdot \vec{\partial}_{23})(\vec{\partial}_{34} \cdot \vec{\partial}_{41}) - (\vec{\partial}_{12} \cdot \vec{\partial}_{34})(\vec{\partial}_{23} \cdot \vec{\partial}_{41}) + (\vec{\partial}_{12} \cdot \vec{\partial}_{41})(\vec{\partial}_{23} \cdot \vec{\partial}_{34})]\}. \end{aligned} \tag{3.44}$$

Combining Eqs. (3.40), (3.41), and (3.44), we can write $W^{(4)}$ in the form

$$\begin{aligned} W^{(4)} = \frac{-i}{2\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^4 2^4 \int \rho_1 d^3x_1 \int \rho_2 d^3x_2 \int \rho_3 d^3x_3 \int \rho_4 d^3x_3 \int_{-\infty}^{\infty} dE \\ \times \mathrm{tr}[\cdots]\Delta_F(\vec{r}_{12}, E)\Delta_F(\vec{r}_{23}, E)\Delta_F(\vec{r}_{34}, E)\Delta_F(\vec{r}_{41}, E), \end{aligned} \tag{3.45}$$

where $\mathrm{tr}[\cdots]$ denotes the expression on the right-hand-side of Eq. (3.44). Using Eq. (B42) the product of the functions $\Delta_F(\vec{r}_{ij}, E)$ can be written as

$$\Delta_F(\vec{r}_{12},E)\Delta_F(\vec{r}_{23},E)\Delta_F(\vec{r}_{34},E)\Delta_F(\vec{r}_{41},E) = \left(\frac{i}{4\pi}\right)^4 \frac{e^{i|E|(r_{12}+r_{23}+r_{34}+r_{41}+i\epsilon)}}{r_{12}r_{23}r_{34}r_{41}}. \tag{3.46}$$

If we denote the sum of the r_{ij} and their product by S_4 and P_4 respectively,

$$S_4 = r_{12} + r_{23} + r_{34} + r_{41}, \tag{3.47a}$$

$$P_4 = r_{12}r_{23}r_{34}r_{41}, \tag{3.47b}$$

then the integral over E can then be expressed in terms of S_4 by using Eqs. (3.11) and (3.14), and then making the replacements

$$E^4 \to 2i\frac{4!}{S_4^5}, \tag{3.48a}$$

$$E^2 \to -2i\frac{2!}{S_4^3}, \tag{3.48b}$$

$$E^0 \to 2i\frac{1}{S_4}. \tag{3.48c}$$

Combining Eqs. (3.45)–(3.47b) we can extract the contribution to the 4-body potential $V^{(4)}$ from diagrams 4(a) and 4(a$'$) by writing

$$W^{(4)}[4(\mathrm{a}) + 4(\mathrm{a}')] = \int \rho_1 d^3x_1 \int \rho_2 d^3x_2 \int \rho_3 d^3x_3 \int \rho_4 d^3x_4 V^{(4)}(\vec{r}_{12},\vec{r}_{23},\vec{r}_{34},\vec{r}_{41}), \tag{3.49}$$

where

$$\begin{aligned} V^{(4)}(\vec{r}_{12},\vec{r}_{23},\vec{r}_{34},\vec{r}_{41}) = \left(\frac{G_F a_n}{\sqrt{2}}\right)^4 \frac{1}{4\pi^5}\Big\{ &\frac{4!}{P_4 S_4^5} \\ &+ 2!\left(\vec{\partial}_{12}\cdot\vec{\partial}_{23} + \vec{\partial}_{23}\cdot\vec{\partial}_{34} + \vec{\partial}_{34}\cdot\vec{\partial}_{41} + \vec{\partial}_{12}\cdot\vec{\partial}_{34} + \vec{\partial}_{12}\cdot\vec{\partial}_{41} + \vec{\partial}_{23}\cdot\vec{\partial}_{41}\right)\frac{1}{P_4 S_4^3} \\ &+ \left[(\vec{\partial}_{12}\cdot\vec{\partial}_{23})(\vec{\partial}_{34}\cdot\vec{\partial}_{41}) - (\vec{\partial}_{12}\cdot\vec{\partial}_{34})(\vec{\partial}_{23}\cdot\vec{\partial}_{41}) + (\vec{\partial}_{12}\cdot\vec{\partial}_{41})(\vec{\partial}_{23}\cdot\vec{\partial}_{34})\right]\frac{1}{P_4 S_4}\Big\}. \end{aligned} \tag{3.50}$$

The expression in Eq. (3.50) reproduces the 4-body result of the Hartle [18] up to some minor misprints in that reference. We note again that the gradient operators $\vec{\partial}_{ij}$ act on P_4 and S_4 as if all the coordinates $\vec{r}_{ij}$ are independent, notwithstanding the fact that $\vec{r}_{12}+\vec{r}_{23}+\vec{r}_{34}+\vec{r}_{41} = 0$.

D. The 6-Body Potential

The general discussion that introduced the 4-body potential in the preceding subsection can be taken over immediately for the 6-body case. Our purpose in deriving the 6-body potential $V^{(6)}$ is to explicitly exhibit the various phases and combinatoric factors in a way that will allow us in the next subsection to obtain the general k-body result $V^{(k)}$. From the preceding discussion we note that for $k = 6$ there are 60 pairs of topologically distinct diagrams, with each pair representing the two possible senses of the neutrino loop momentum. The 6 legs of the standard diagram correspond to the variables $\vec{r}_{12}, \vec{r}_{23}, \vec{r}_{34}, \vec{r}_{45}, \vec{r}_{56}$, and $\vec{r}_{61}$ which satisfy

$$\vec{r}_{12} + \vec{r}_{23} + \vec{r}_{34} + \vec{r}_{45} + \vec{r}_{56} + \vec{r}_{61} = 0, \tag{3.51}$$

and S_4, P_4 in Eqs. (3.47a) and (3.47b) are now replaced by

$$S_6 = r_{12} + r_{23} + r_{34} + r_{45} + r_{56} + r_{61}, \tag{3.52a}$$

$$P_6 = r_{12} r_{23} r_{34} r_{45} r_{56} r_{61}. \tag{3.52b}$$

Using the formalism of the previous subsection the expression for $V^{(6)}(\vec{r}_{12}, \ldots, \vec{r}_{61})$ can be written as follows:

$$V^{(6)}(\vec{r}_{12}, \ldots, \vec{r}_{61}) = \frac{-i}{2\pi} \left(\frac{i}{4\pi}\right)^6 (i)^7 2^7 \left(\frac{G_F a_n}{\sqrt{2}}\right)^6 4\Phi^{(6)}(S_6, P_6), \tag{3.53}$$

where

$$\begin{aligned}
\Phi^{(6)}(S_6, P_6) = \frac{(i)^6 6!}{P_6 S_6^7} &- (i)^4 4! \Big[\vec{\partial}_{12} \cdot \vec{\partial}_{23} + \vec{\partial}_{34} \cdot \vec{\partial}_{45} + \vec{\partial}_{56} \cdot \vec{\partial}_{61} + \vec{\partial}_{12} \cdot \vec{\partial}_{34} + \vec{\partial}_{23} \cdot \vec{\partial}_{45} \\
&+ \vec{\partial}_{12} \cdot \vec{\partial}_{45} + \vec{\partial}_{23} \cdot \vec{\partial}_{34} - \vec{\partial}_{12} \cdot \vec{\partial}_{56} - \vec{\partial}_{23} \cdot \vec{\partial}_{61} - \vec{\partial}_{12} \cdot \vec{\partial}_{61} \\
&- \vec{\partial}_{23} \cdot \vec{\partial}_{56} + \vec{\partial}_{34} \cdot \vec{\partial}_{61} + \vec{\partial}_{34} \cdot \vec{\partial}_{56} + \vec{\partial}_{45} \cdot \vec{\partial}_{61} + \vec{\partial}_{45} \cdot \vec{\partial}_{56}\Big] \frac{1}{P_6 S_6^5} \\
&+ (i)^2 2! \Big[(\vec{\partial}_{12} \cdot \vec{\partial}_{23})(\vec{\partial}_{45} \cdot \vec{\partial}_{56}) + (\vec{\partial}_{12} \cdot \vec{\partial}_{23})(\vec{\partial}_{34} \cdot \vec{\partial}_{61}) + (\vec{\partial}_{34} \cdot \vec{\partial}_{61})(\vec{\partial}_{45} \cdot \vec{\partial}_{56}) \\
&+ (\vec{\partial}_{12} \cdot \vec{\partial}_{23})(\vec{\partial}_{34} \cdot \vec{\partial}_{56}) + (\vec{\partial}_{12} \cdot \vec{\partial}_{23})(\vec{\partial}_{45} \cdot \vec{\partial}_{61}) - (\vec{\partial}_{34} \cdot \vec{\partial}_{56})(\vec{\partial}_{45} \cdot \vec{\partial}_{61}) \\
&- (\vec{\partial}_{12} \cdot \vec{\partial}_{34})(\vec{\partial}_{23} \cdot \vec{\partial}_{61}) + (\vec{\partial}_{12} \cdot \vec{\partial}_{34})(\vec{\partial}_{45} \cdot \vec{\partial}_{56}) + (\vec{\partial}_{23} \cdot \vec{\partial}_{61})(\vec{\partial}_{45} \cdot \vec{\partial}_{56})
\end{aligned}$$

$$
\begin{aligned}
&+(\vec{\partial}_{12}\cdot\vec{\partial}_{34})(\vec{\partial}_{45}\cdot\vec{\partial}_{61})-(\vec{\partial}_{12}\cdot\vec{\partial}_{34})(\vec{\partial}_{23}\cdot\vec{\partial}_{56})-(\vec{\partial}_{23}\cdot\vec{\partial}_{56})(\vec{\partial}_{45}\cdot\vec{\partial}_{61})\\
&+(\vec{\partial}_{23}\cdot\vec{\partial}_{61})(\vec{\partial}_{34}\cdot\vec{\partial}_{56})-(\vec{\partial}_{12}\cdot\vec{\partial}_{45})(\vec{\partial}_{23}\cdot\vec{\partial}_{61})-(\vec{\partial}_{12}\cdot\vec{\partial}_{45})(\vec{\partial}_{34}\cdot\vec{\partial}_{56})\\
&-(\vec{\partial}_{12}\cdot\vec{\partial}_{45})(\vec{\partial}_{23}\cdot\vec{\partial}_{56})-(\vec{\partial}_{12}\cdot\vec{\partial}_{45})(\vec{\partial}_{34}\cdot\vec{\partial}_{61})-(\vec{\partial}_{23}\cdot\vec{\partial}_{56})(\vec{\partial}_{34}\cdot\vec{\partial}_{61})\\
&+(\vec{\partial}_{23}\cdot\vec{\partial}_{34})(\vec{\partial}_{12}\cdot\vec{\partial}_{61})+(\vec{\partial}_{23}\cdot\vec{\partial}_{34})(\vec{\partial}_{45}\cdot\vec{\partial}_{56})+(\vec{\partial}_{12}\cdot\vec{\partial}_{61})(\vec{\partial}_{45}\cdot\vec{\partial}_{56})\\
&+(\vec{\partial}_{23}\cdot\vec{\partial}_{34})(\vec{\partial}_{12}\cdot\vec{\partial}_{56})+(\vec{\partial}_{23}\cdot\vec{\partial}_{34})(\vec{\partial}_{45}\cdot\vec{\partial}_{61})-(\vec{\partial}_{12}\cdot\vec{\partial}_{56})(\vec{\partial}_{45}\cdot\vec{\partial}_{61})\\
&+(\vec{\partial}_{12}\cdot\vec{\partial}_{61})(\vec{\partial}_{34}\cdot\vec{\partial}_{56})-(\vec{\partial}_{23}\cdot\vec{\partial}_{45})(\vec{\partial}_{34}\cdot\vec{\partial}_{56})+(\vec{\partial}_{23}\cdot\vec{\partial}_{45})(\vec{\partial}_{12}\cdot\vec{\partial}_{61})\\
&+(\vec{\partial}_{23}\cdot\vec{\partial}_{45})(\vec{\partial}_{12}\cdot\vec{\partial}_{56})-(\vec{\partial}_{23}\cdot\vec{\partial}_{45})(\vec{\partial}_{34}\cdot\vec{\partial}_{61})-(\vec{\partial}_{12}\cdot\vec{\partial}_{56})(\vec{\partial}_{34}\cdot\vec{\partial}_{61})\\
&-(\vec{\partial}_{34}\cdot\vec{\partial}_{45})(\vec{\partial}_{23}\cdot\vec{\partial}_{56})-(\vec{\partial}_{34}\cdot\vec{\partial}_{45})(\vec{\partial}_{12}\cdot\vec{\partial}_{61})-(\vec{\partial}_{12}\cdot\vec{\partial}_{61})(\vec{\partial}_{23}\cdot\vec{\partial}_{56})\\
&+(\vec{\partial}_{12}\cdot\vec{\partial}_{56})(\vec{\partial}_{23}\cdot\vec{\partial}_{61})-(\vec{\partial}_{34}\cdot\vec{\partial}_{45})(\vec{\partial}_{12}\cdot\vec{\partial}_{56})-(\vec{\partial}_{34}\cdot\vec{\partial}_{45})(\vec{\partial}_{23}\cdot\vec{\partial}_{61})\\
&+(\vec{\partial}_{12}\cdot\vec{\partial}_{45})(\vec{\partial}_{23}\cdot\vec{\partial}_{34})+(\vec{\partial}_{12}\cdot\vec{\partial}_{45})(\vec{\partial}_{56}\cdot\vec{\partial}_{61})+(\vec{\partial}_{23}\cdot\vec{\partial}_{34})(\vec{\partial}_{56}\cdot\vec{\partial}_{61})\\
&+(\vec{\partial}_{23}\cdot\vec{\partial}_{45})(\vec{\partial}_{56}\cdot\vec{\partial}_{61})-(\vec{\partial}_{12}\cdot\vec{\partial}_{34})(\vec{\partial}_{23}\cdot\vec{\partial}_{45})+(\vec{\partial}_{12}\cdot\vec{\partial}_{34})(\vec{\partial}_{56}\cdot\vec{\partial}_{61})\\
&\left.+(\vec{\partial}_{12}\cdot\vec{\partial}_{23})(\vec{\partial}_{34}\cdot\vec{\partial}_{45})+(\vec{\partial}_{12}\cdot\vec{\partial}_{23})(\vec{\partial}_{56}\cdot\vec{\partial}_{61})+(\vec{\partial}_{34}\cdot\vec{\partial}_{45})(\vec{\partial}_{56}\cdot\vec{\partial}_{61})\right]\frac{1}{P_6S_6^3}\\
&+\Big[-(\vec{\partial}_{12}\cdot\vec{\partial}_{23})(\vec{\partial}_{34}\cdot\vec{\partial}_{61})(\vec{\partial}_{45}\cdot\vec{\partial}_{56})+(\vec{\partial}_{12}\cdot\vec{\partial}_{23})(\vec{\partial}_{34}\cdot\vec{\partial}_{56})(\vec{\partial}_{45}\cdot\vec{\partial}_{61})\\
&+(\vec{\partial}_{12}\cdot\vec{\partial}_{34})(\vec{\partial}_{23}\cdot\vec{\partial}_{61})(\vec{\partial}_{45}\cdot\vec{\partial}_{56})-(\vec{\partial}_{12}\cdot\vec{\partial}_{34})(\vec{\partial}_{23}\cdot\vec{\partial}_{56})(\vec{\partial}_{45}\cdot\vec{\partial}_{61})\\
&-(\vec{\partial}_{12}\cdot\vec{\partial}_{45})(\vec{\partial}_{23}\cdot\vec{\partial}_{61})(\vec{\partial}_{34}\cdot\vec{\partial}_{56})+(\vec{\partial}_{12}\cdot\vec{\partial}_{45})(\vec{\partial}_{23}\cdot\vec{\partial}_{56})(\vec{\partial}_{34}\cdot\vec{\partial}_{61})\\
&-(\vec{\partial}_{23}\cdot\vec{\partial}_{34})(\vec{\partial}_{12}\cdot\vec{\partial}_{61})(\vec{\partial}_{45}\cdot\vec{\partial}_{56})+(\vec{\partial}_{23}\cdot\vec{\partial}_{34})(\vec{\partial}_{12}\cdot\vec{\partial}_{56})(\vec{\partial}_{45}\cdot\vec{\partial}_{61})\\
&+(\vec{\partial}_{23}\cdot\vec{\partial}_{45})(\vec{\partial}_{12}\cdot\vec{\partial}_{61})(\vec{\partial}_{34}\cdot\vec{\partial}_{56})-(\vec{\partial}_{23}\cdot\vec{\partial}_{45})(\vec{\partial}_{12}\cdot\vec{\partial}_{56})(\vec{\partial}_{34}\cdot\vec{\partial}_{61})\\
&+(\vec{\partial}_{34}\cdot\vec{\partial}_{45})(\vec{\partial}_{12}\cdot\vec{\partial}_{61})(\vec{\partial}_{23}\cdot\vec{\partial}_{56})-(\vec{\partial}_{34}\cdot\vec{\partial}_{45})(\vec{\partial}_{12}\cdot\vec{\partial}_{56})(\vec{\partial}_{23}\cdot\vec{\partial}_{61})\\
&-(\vec{\partial}_{12}\cdot\vec{\partial}_{45})(\vec{\partial}_{23}\cdot\vec{\partial}_{34})(\vec{\partial}_{56}\cdot\vec{\partial}_{61})+(\vec{\partial}_{12}\cdot\vec{\partial}_{34})(\vec{\partial}_{23}\cdot\vec{\partial}_{45})(\vec{\partial}_{56}\cdot\vec{\partial}_{61})\\
&\left.-(\vec{\partial}_{12}\cdot\vec{\partial}_{23})(\vec{\partial}_{34}\cdot\vec{\partial}_{45})(\vec{\partial}_{56}\cdot\vec{\partial}_{61})\right]\frac{1}{P_6S_6}
\end{aligned}
\tag{3.54}
$$

It is helpful to understand the sources of the various numerical factors in $V^{(6)}$ as a prelude to discussing the k-body potential $V^{(k)}$. The overall coefficient $(-i/2\pi)$ arises from Eq. (B34), with an additional minus sign from the expansion of $\ln(1+\Delta)$ in Eq. (3.2). Each of the neutrino propagators in Eq. (B40) contributes a factor $(i/4\pi)$, and $I_n(z)$ in Eq. (3.14) is responsible for the factor $(i)^7$. The 6 factors of $(1+\gamma_5)$ eventually collapse to a single factor

by use of Eq. (3.4), leaving a numerical coefficient of 2^5. An additional factor of 2 arises from the diagram with the reversed loop momentum and this, along with the factor of 2 from $I_n(z)$ in Eq. (3.14), explains the factor 2^7. The Dirac trace leads to the coefficient 4 multiplying $\Phi^{(6)}$, and in 6th order the overall strength of the 6-body interaction is given by $(G_F a_n/\sqrt{2})^6$.

E. The k-Body Potential

The preceding discussion can be generalized in a straightforward manner to the k-body potential $V^{(k)}$. As we discuss in Sec. V below, it is sufficient for present purposes to consider only the single term in $V^{(k)}$ which is independent of derivatives, and hence we wish to exhibit this term in complete detail. We will also exhibit the coefficients of the derivative terms, whose forms can be obtained by generalizing Eq. (3.53) in an obvious way. Collecting together the various powers of i and factors of 2, we can write the contribution to $V^{(k)}$ from the standard diagram in $\mathcal{O}(G_F^k)$ as follows:

$$V^{(k)}(\vec{r}_{12}, \ldots, \vec{r}_{k1}) = \frac{1}{2\pi}(i)^{2k} 2^{k+1} \left(\frac{1}{4\pi}\right)^k \left(\frac{G_F a_n}{\sqrt{2}}\right)^k 4\Phi^{(k)}(S_k, P_k), \tag{3.55}$$

where

$$\begin{aligned}\Phi^{(k)}(S_k, P_k) = {} & \frac{(i)^k k!}{P_k S_k^{k+1}} \\ & + (i)^{k-2}(k-2)! \left[\sum\nolimits^{(2)} (\vec{\partial}_{ab} \cdot \vec{\partial}_{rs})\right] \frac{1}{P_k S_k^{k-1}} \\ & + (i)^{k-4}(k-4)! \left[\sum\nolimits^{(4)} (\vec{\partial}_{ab} \cdot \vec{\partial}_{rs})(\vec{\partial}_{cd} \cdot \vec{\partial}_{lm})\right] \frac{1}{P_k S_k^{k-3}} \\ & + \cdots + \left[\sum\nolimits^{(k)} (\vec{\partial}_{ab} \cdot \vec{\partial}_{rs})(\vec{\partial}_{cd} \cdot \partial_{lm}) \cdots\right] \frac{1}{P_k S_k},\end{aligned} \tag{3.56}$$

$$S_k = r_{12} + r_{23} + r_{34} + \cdots + r_{k1}, \tag{3.57}$$

$$P_k = r_{12} r_{23} r_{34} \cdots r_{k1}. \tag{3.58}$$

In Eq. (3.55) the notation $\sum^{(2)}, \sum^{(4)}, \ldots, \sum^{(k)}$ symbolically represents the terms containing products of $2, 4, \ldots, k$ derivatives, which act to the right on the indicated functions of P_k and

S_k. The numerical coefficient of $\Phi^{(k)}$ can be understood in light of the preceding discussion as follows: The factor of $1/2\pi$ arises from the product of the overall coefficient $(-i/2\pi)$ of W in Eq. (B34), Eq. (3.2), and a factor of i extracted from $I_n(z)$ in Eq. (3.14), where $n \to k$, $k-2$, $k-4, \ldots$ as appropriate in $\Phi^{(k)}$. The factor of 2^{k+1} is the product of 2^{k-1} arising from k factors of $(1+\gamma_5)$, and factors of 2 from $I_n(z)$ and from the diagram with the reversed neutrino-loop momentum. The overall strength of the k-body contribution is determined by $(G_F a_n/\sqrt{2})^k$, and the neutrino propagators contribute the factor $(1/4\pi)^k$ and the phase $(i)^k$. This phase, when combined with the k powers of i resulting from the generalization of $T_{\mu\nu\lambda\sigma}(x_1, \ldots, x_4)$ in Eq. (3.41), leads to the overall phase $(i)^{2k} = +1$ for even k. As before, the Dirac trace contributes the coefficient 4 of $\Phi^{(k)}$, and $I_n(z)$ gives the coefficients $k!, (k-2)!, \ldots$ of the individual terms in $\Phi^{(k)}$. Collecting together the various factors in Eq. (3.55) we have (for k even)

$$V^{(k)}(\vec{r}_{12}, \ldots, \vec{r}_{k1}) = \frac{4}{\pi}\left(\frac{G_F a_n}{2\pi\sqrt{2}}\right)^k \Phi^{(k)}(S_k, P_k). \tag{3.59}$$

It is worth noting that the coefficient of $\Phi^{(k)}$ is real, as is each of the terms in $\Phi^{(k)}$. However, because the individual phase factors $(i)^k, (i)^{k-2}, \ldots$ can (depending on k) assume the values (± 1), various partial cancellations take place. Since the individual terms are unphysically large, these cancellations do not eliminate the problem arising from the energy-density due to neutrino exchange, but produce other novel unphysical effects, as we discuss in Sec. V below. We will thus be led eventually to the conclusion that because all these effects are consequences of the assumption that neutrinos are massless, there must in fact be a lower bound on the mass of any neutrino.

IV. INTEGRATION OF THE NEUTRINO-EXCHANGE POTENTIALS OVER A SPHERICAL VOLUME

Having derived the analytic forms of the k-body potentials in the previous section, we turn to the problem of integrating these over a spherical volume, in analogy to the electrostatic case.

A. The 2-Body Potential

From Eq. (3.21) the 2-body potential between neutrons arising from neutrino exchange is

$$V^{(2)}(r) = \frac{(G_F a_n)^2}{4\pi^3 \hbar c}\frac{1}{r^5} \equiv \frac{\kappa}{r^5}, \tag{4.1}$$

where r is the distance between the neutrons. In the absence of a cutoff limiting how small r can become, the integral of $V^{(2)}(r)$ over a spherical volume would be singular. This contrasts with the electrostatic case discussed in Sec. II, where the radial factors in the volume element $d^3x_1 d^3x_2$ in Eq. (2.5) offset the contribution from the potential e_0^2/r_{12} and lead to a well-behaved result. In practice a natural cutoff on r exists in the systems of interest to us, specifically in nuclei and in neutron stars. In the former case it is well known [30] that the nucleon-nucleon interaction has a strong repulsive component (the "hard core") which prevents the nucleon-nucleon separation r_c from becoming smaller than approximately 0.5 fm. The dynamics of neutron stars, although less well known, would also require a hard core. Combining Eq. (4.1) with Eqs. (2.10) and (2.12) we find for the 2-body contribution $U^{(2)}$ arising from neutrino exchange

$$\begin{aligned} U^{(2)} &= \int_{r_c}^{2R} dr\, \mathcal{P}(r) V^{(2)}(r) \\ &= \int_{r_c}^{2R} dr\, \mathcal{P}(r) \left(\frac{\kappa}{r^5}\right) \\ &= \kappa \left[\frac{3}{2R^3 r_c^2}\left(1 - \frac{r_c^2}{4R^2}\right) - \frac{9}{4R^4 r_c}\left(1 - \frac{r_c}{2R}\right) + \frac{3}{16R^6}(2R - r_c)\right] \\ &\simeq \frac{3\kappa}{2}\frac{1}{R^3 r_c^2}. \end{aligned} \tag{4.2}$$

Eq. (4.2), which is the analog of Eq. (2.5) in the electrostatic case, gives the average interaction energy for a single pair of neutrons having a uniform probability distribution in a spherical volume of radius R.

It is worth commenting on the implication of the fact that the integration of $V^{(2)}(r)$ over the spherical volume leads to the introduction of an additional dimensional parameter, namely r_c. We note to start with that since $r_c < R$ the presence of r_c *strengthens* the

dimensional arguments given in the Introduction concerning the approximate magnitude of the contributions from neutrino exchange. Secondly, we anticipate the ensuing discussion of the k-body contributions by noting that for $k \geq 4$ the integrations of the potentials $V^{(k)}$ lead to expressions for $U^{(k)}$ which are well-behaved as $r_c \to 0$. Hence with the exception of $U^{(2)}$, which makes a negligible contribution to the total energy density, all $U^{(k)}$ for $k \geq 4$ depend only on G_F and R, as noted in the Introduction.

B. The 4-Body Potential

The new feature which arises in the integration of $V^{(4)}(r_{12}, \ldots, r_{41})$, as given in Eqs. (3.49) and (3.50), is the presence of angular factors which result from the scalar products of the derivative terms. Although one might be tempted to assume that any such factors average to zero when integrated over a sphere, this is not the case as we discuss below. The evaluation of the terms involving derivatives can be simplified by generalizing the result in Eq. (3.17). To start with we recall from Eq. (B42) that all the derivatives have their origin in the neutrino propagator $S_F^{(0)}$, where $\vec{\partial}_{ij}$ acts on r_{ij} appearing in $\Delta_F(\vec{r}_{ij}, E)$. In principle the presence of all derivatives can be eliminated at the outset by evaluating $\vec{\partial}_{ij}\Delta_F(\vec{r}_{ij}, E)$ before carrying out $\int dE$. Although it is more convenient to evaluate $\int dE$ first, one must recall that the $\vec{\partial}_{ij}$ which then appear in $V^{(4)}$ are to be understood as acting on the corresponding $\vec{r}_{ij}$ as if all the $\vec{r}_{ij}$ were independent. It is straightforward to show, for example, that the same 2-body result is obtained if one evaluates $\vec{\partial}_{ij}\Delta_F$ first, rather than leaving the derivatives to the end as was done in arriving at Eq. (3.16).

Since a derivative such as $\vec{\partial}_{12}$ in $V^{(4)}$ acts on a function $f(r_{12}, \ldots)$ which depends only on the magnitude of $\vec{r}_{12}$ (but not its direction) we can write

$$\frac{\partial}{\partial \vec{r}_{12}} f(r_{12}, \ldots) = \hat{r}_{12} \frac{\partial}{\partial r_{12}} f(r_{12}, \ldots). \tag{4.3}$$

It follows that

$$\vec{\partial}_{12} \cdot \vec{\partial}_{23} f(r_{12}, r_{23}, \ldots) \equiv \frac{\partial}{\partial \vec{r}_{12}} \cdot \frac{\partial}{\partial \vec{r}_{23}} f(r_{12}, r_{23}, \ldots) = \hat{r}_{12} \cdot \hat{r}_{23} \frac{\partial}{\partial r_{12}} \frac{\partial}{\partial r_{23}} f(r_{12}, r_{23}, \ldots). \tag{4.4}$$

Unlike the 2-body case, where the analog of Eq. (4.4) simplifies because $\hat{r}_{12} \cdot \hat{r}_{21} = -1$, the angular factor $\hat{r}_{12} \cdot \hat{r}_{23}$ in Eq. (4.4) is not a constant, and does not average to zero when integrated over a sphere. Moreover, the expression in Eq. (4.4) cannot be calculated by separately evaluating the angular factor and the function which it multiplies, since these contributions are not independent.

To further explore the angular factors we consider $\langle \hat{r}_{ij} \cdot \hat{r}_{lm} \rangle_R$, where the notation $\langle \cdots \rangle_R$ denotes the average over a spherical volume of radius R, and $i, j, l, m = 1, \ldots, 4$. $\langle \hat{r}_{12} \cdot \hat{r}_{34} \rangle_R$ and $\langle \hat{r}_{12} \cdot \hat{r}_{23} \rangle_R$ have been evaluated numerically [35] by randomly generating 10^8 sets of vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, and $\vec{r}_4$ in a sphere of radius $R = 0.5$. We find

$$\langle \hat{r}_{12} \cdot \hat{r}_{34} \rangle_R = \left\langle \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} \cdot \frac{(\vec{r}_3 - \vec{r}_4)}{|\vec{r}_3 - \vec{r}_4|} \right\rangle_R \simeq 0, \tag{4.5a}$$

$$\langle \hat{r}_{12} \cdot \hat{r}_{23} \rangle_R = \left\langle \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} \cdot \frac{(\vec{r}_2 - \vec{r}_3)}{|\vec{r}_2 - \vec{r}_3|} \right\rangle_R = -0.442. \tag{4.5b}$$

The result in Eq. (4.5a), which is zero to within the expected statistical fluctuations, can be understood as follows: Given a sufficient number of trials (i.e., sets of vectors) we would find that for each configuration of 4 particles specified by $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, and $\vec{r}_4$ there will be another in which the coordinates of particles 1 and 2 are same but those of particles 3 and 4 are interchanged. The contributions from these two configurations to $\langle \hat{r}_{12} \cdot \hat{r}_{34} \rangle_R$ evidently cancel, which accounts for Eq. (4.5a). By contrast, the result in Eq. (4.5b) is less obvious, but the fact that $\langle \hat{r}_{12} \cdot \hat{r}_{23} \rangle_R$ is nonzero can be understood as follows: To start with the preceding argument cannot be applied to $\langle \hat{r}_{12} \cdot \hat{r}_{23} \rangle_R$ since changing the coordinates of particle 2 affects both $\hat{r}_{12}$ and $\hat{r}_{23}$. That such a scalar product is non-zero when averaged over the sphere can be understood if we replace the unit vectors by the corresponding dimensional vectors, $\hat{r}_{ij} \rightarrow \vec{r}_{ij}$, etc. Then

$$\langle \vec{r}_{12} \cdot \vec{r}_{23} \rangle_R = \langle (\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_2 - \vec{r}_3) \rangle_R = \langle \vec{r}_1 \cdot \vec{r}_2 - \vec{r}_1 \cdot \vec{r}_3 + \vec{r}_2 \cdot \vec{r}_3 - r_2^2 \rangle_R. \tag{4.6}$$

The $\vec{r}_i \cdot \vec{r}_j$ terms in Eq. (4.6) average to zero by an argument similar to that for $\langle \hat{r}_{12} \cdot \hat{r}_{34} \rangle_R$: Given a sufficient number of trials, then for each random pair of vectors $\vec{r}_i$ and $\vec{r}_j$, there

will be another in which $\vec{r}_i$ will be the same, but the coordinates of r_j will be inverted $(\vec{r}_j \to -\vec{r}_j)$. The sums of these two contributions evidently cancel, which leaves

$$\langle \vec{r}_{12} \cdot \vec{r}_{23} \rangle_R = \langle -r_2^2 \rangle_R = -\frac{\int_0^R r_2^2 \cdot 4\pi r_2^2 dr_2}{\int_0^R 4\pi r_2^2 dr_2} = -\frac{3}{5} R^2. \tag{4.7}$$

We note that since $\vec{r}_2$ is the coordinate of particle 2 measured from the center of the sphere, the upper limit on the r_2 integration is R and not $2R$, as in Eq. (4.2). (In the 2-body case considered in the previous subsection r in Eq. (4.2) denotes the distance between two points whose maximum value is $2R$.) The analytic result in Eq. (4.7) has been verified numerically, and was used as a check on other numerical results.

We conclude from the above that at least some of the angular-dependent factors in $V^{(k)}$ for $k \geq 4$ are non-zero. One can show, moreover, that even an angular factor such as $\hat{r}_{12} \cdot \hat{r}_{34}$, which averages to zero over the sphere when considered by itself, can give a non-zero contribution when multiplied by the functions which arise from differentiating S_k and P_k in Eq. (3.50). To see this consider the term in Eq. (3.50) proportional to $\hat{r}_{12} \cdot \hat{r}_{34}$:

$$\begin{aligned} \vec{\partial}_{12} \cdot \vec{\partial}_{34} \frac{1}{P_4 S_4^3} &= \hat{r}_{12} \cdot \hat{r}_{34} \frac{\partial}{\partial r_{12}} \frac{\partial}{\partial r_{34}} \left[\frac{1}{r_{12} r_{23} r_{34} r_{41} (r_{12} + r_{23} + r_{34} + r_{41})^3} \right] \\ &= \hat{r}_{12} \cdot \hat{r}_{34} \\ &\quad \times \left[\frac{4(r_{12}^2 + r_{34}^2) + r_{23}^2 + r_{41}^2 + 5(r_{12} r_{23} + r_{23} r_{34} + r_{34} r_{41} + r_{41} r_{12}) + 20 r_{12} r_{34} + 2 r_{23} r_{41}}{r_{12}^2 r_{23} r_{34}^2 r_{41} (r_{12} + r_{23} + r_{34} + r_{41})^5} \right]. \end{aligned} \tag{4.8}$$

Returning to the discussion following Eq. (4.5b) we note that the contributions from two configurations which differ by the interchange of the coordinates of particles 3 and 4 no longer cancel in Eq. (4.8). This is because such an interchange effectively replaces the right-hand side of Eq. (4.8) by

$$-\hat{r}_{12} \cdot \hat{r}_{34} \left[\frac{4(r_{12}^2 + r_{34}^2) + r_{24}^2 + r_{31}^2 + 5(r_{12} r_{24} + r_{24} r_{34} + r_{34} r_{13} + r_{13} r_{12}) + 20 r_{12} r_{34} + 2 r_{24} r_{13}}{r_{12}^2 r_{24} r_{34}^2 r_{13} (r_{12} + r_{24} + r_{34} + r_{13})^5} \right]. \tag{4.9}$$

With the coordinates $\vec{r}_1$ and $\vec{r}_2$ remaining fixed, the expressions multiplying $\hat{r}_{12} \cdot \hat{r}_{34}$ in Eqs. (4.8) and (4.9) are not equal in general. (The exception would be a 4-body configuration

in which the coordinates $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, and $\vec{r}_4$ formed a regular tetrahedron. However, in 3-dimensional space such a configuration is not possible for more than 4-bodies.) The above argument has been verified numerically by a Monte Carlo simulation, as we discuss in more detail elsewhere [35].

It follows from the preceding discussion that all of the terms in Eq. (3.50) are expected to contribute to $U^{(4)}$, and this has been verified numerically by means of a Monte Carlo simulation [35]. Briefly, the symbolic program MATHEMATICA [36] was used at the outset to explicitly evaluate all the derivatives appearing in Eq. (3.50), as was done in Eq. (4.8) and (4.9). The resulting expression for $V^{(4)}(\vec{r}_{12}, \vec{r}_{23}, \vec{r}_{34}, \vec{r}_{41})$ was then evaluated for each of 10^8 configurations, where a configuration is obtained by randomly generating a set of four 3-vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, and $\vec{r}_4$. If $V_i^{(4)}(\vec{r}^{\,i}_{12}, \vec{r}^{\,i}_{23}, \vec{r}^{\,i}_{34}, \vec{r}^{\,i}_{41})$ denotes the value of $V^{(4)}$ obtained from the ith configuration, then

$$U^{(4)} = \frac{1}{N_r} \sum_{i=1}^{N_r} V_i^{(4)}(\vec{r}^{\,i}_{12}, \vec{r}^{\,i}_{23}, \vec{r}^{\,i}_{34}, \vec{r}^{\,i}_{41}) \tag{4.10}$$

where $N_r = 10^8$ in our calculations. We find numerically [35],

$$U^{(4)} = \frac{4}{\pi R} \left(\frac{G_F a_n}{2\sqrt{2}\pi R^2} \right)^4 (7.7). \tag{4.11}$$

There are several implications of the 4-body results which will be useful in the ensuing discussion. To start with there is no theoretical reason why $U^{(4)}$ should vanish, and there is no suggestion that it does either analytically or numerically. Secondly, the computational effort required to evaluate $U^{(4)}$ is sufficiently great as to question the feasibility of a calculation of even $U^{(6)}$ and $U^{(8)}$, let alone $U^{(k)}$ for $k = \mathcal{O}(10^{57})$. We note that a calculation of $U^{(k)}$ starting from a generalized version of Eq. (2.4) would require a $3k$-dimensional integral, and hence this approach is impractical for large k. Fortunately it turns out that for present purposes it suffices to determine the 0-derivative contributions $U_0^{(k)}$ to $U^{(k)}$, for which we can obtain an approximate analytic result as we describe below.

Having evaluated $U^{(4)}$ directly by a Monte-Carlo simulation, we ask whether it is possible to generalize the probability density function $\mathcal{P}(r)$ in Eq. (2.12) to the 4-body case. If this

were possible it would allow the Monte-Carlo evaluation to be carried out *analytically*, just as in the 2-body case. A major obstacle that must be confronted in any attempt to generalize $\mathcal{P}(r)$ is that the appropriate k-body probability density function must depend not only on the separations $r_{ij} = |\vec{r}_i - \vec{r}_j|$ of i and j, but also on angular factors such as $\hat{r}_{ij} \cdot \hat{r}_{lm}$ which arise from the derivative terms. Even for the 0-derivative term the generalization of $\mathcal{P}(r)$ to the k-body case is not known at present, as we discuss in Appendix C. For this reason, we develop in Appendix C the "mean value approximation" which allows the k-body integrals to be done analytically, and which we use for $k \geq 6$.

C. The k-Body Potential for $k \geq 6$

As we have noted previously, $W = \sum_k W^{(k)} = \sum_k U^{(k)} \binom{N}{k}$ in Eq. (B1) is dominated by terms with $k \simeq N = \mathcal{O}(10^{57})$. For values of k this large, numerical evaluation of $U^{(k)}$ is not possible, even for the 0-derivative terms. However, there is a useful bound on the 0-derivative contribution $U_0^{(k)}$ to $U^{(k)}$ which we derive in this subsection which, along with the "mean value approximation" in Appendix C, forms the basis for the ensuing discussion. Before doing so we present the numerical result for $U_0^{(6)}$ for illustrative purposes,

$$U_0^{(6)} = -\frac{4}{\pi R} \left(\frac{G_F a_n}{2\sqrt{2}\pi R^2} \right)^6 C_6, \tag{4.12}$$

$$C_6 = 0.268(34). \tag{4.13}$$

We proceed to derive the previously-discussed bound on $U_0^{(k)}$. Let $\mathcal{P}^{(k)}(r_{12}, r_{23}, \ldots, r_{k1})$ denote the probability density for the separation of particles 1 and 2 to be in the interval between r_{12} and $r_{12} + dr_{12}$, etc. Thus, the differential probability $d^k\Pi^{(k)}$ of a configuration specified by $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_k$ is

$$d^k\Pi^{(k)} = \mathcal{P}^{(k)}(r_{12}, r_{23}, \ldots, r_{k1}) dr_{12} dr_{23} \cdots dr_{k1}, \tag{4.14}$$

where

$$\int_0^{2R} dr_{12} \cdots \int_0^{2R} dr_{k1} \mathcal{P}^{(k)}(r_{12}, r_{23}, \ldots, r_{k1}) = 1. \tag{4.15}$$

We know that the function $\mathcal{P}^{(k)}(r_{12}, r_{23}, \ldots, r_{k1})$ exists because it can be determined numerically by randomly generating sets of vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_k$ in a sphere, and then computing the probability $d^k\Pi^{(k)}$ of any configuration, as in Eq. (4.14). In fact the first non-trivial generalization of $\mathcal{P}(r)$ in Eq. (2.12), namely $\mathcal{P}^{(3)}(r_{12}, r_{23}, r_{31})$, has been inferred in exactly this way [35]. (As noted in Sec. III, the fact that there is no 3-body contribution to W has to do with the functional form of $V^{(3)}$, not $\mathcal{P}^{(3)}(r_{12}, r_{23}, r_{31})$.) Combining Eqs. (4.14) and (3.55) we find:

$$\begin{aligned} U_0^{(k)} = \frac{4}{\pi} \left(\frac{G_F a_n}{2\pi\sqrt{2}} \right)^k i^k k! \int_0^{2R} dr_{12} \cdots \int_0^{2R} dr_{k1} \mathcal{P}^{(k)}(r_{12}, r_{23}, \ldots, r_{k1}) \\ \times \left[\frac{1}{r_{12} r_{23} \cdots r_{k1} (r_{12} + r_{23} + \cdots + r_{k1})^{k+1}} \right]. \end{aligned} \tag{4.16}$$

We observe that the function multiplying $\mathcal{P}^{(k)}(\cdots)$ is non-negative, and is a monotonically decreasing function of r_{ij} in the interval $[0, 2R]$. It follows that the minimum value of $|U_0^{(k)}|$ is achieved when $r_{ij} = 2R$ for all i, j. Thus,

$$|U_0^{(k)}| \geq \frac{4}{\pi} \left(\frac{G_F a_n}{2\pi\sqrt{2}} \right)^k k! \frac{1}{(2R)^k (2Rk)^{k+1}} \int_0^{2R} dr_{12} \cdots \int_0^{2R} dr_{k1} \mathcal{P}^{(k)}(r_{12}, r_{23}, \ldots, r_{k1}). \tag{4.17}$$

Using the normalization condition, Eq. (4.15), we have

$$|U_0^{(k)}| \geq \frac{2}{\pi R} \left(\frac{G_F a_n}{8\pi\sqrt{2} R^2} \right)^k \frac{k!}{k^{k+1}}. \tag{4.18}$$

The result in Eq. (4.18) represents the $\mathcal{O}(G_F^k)$ contribution from the standard diagram (with both senses of the loop momentum included) as can be seen from the starting point in Eq. (3.55). As noted in Section III and in Fig. 4, there are in $\mathcal{O}(G_F^k)$ a total of $(k-1)!/2$ topologically distinct pairs of diagrams, with each pair representing the sum of the contributions from both senses of the loop momentum. The sum of all these $(k - 1)!/2$ diagrams replaces the expression in square brackets in Eq. (4.16) by the appropriately symmetrized [in the $k(k-1)/2$ variables r_{ij}] generalization of the standard contribution exhibited there. Since $\mathcal{P}^{(k)}(r_{12}, r_{23}, \ldots, r_{k1})$ is also a symmetric function of r_{ij}, we conclude that each of the

$(k-1)!/2$ contributions to $U_0^{(k)}$ will be equal in magnitude to that arising from Eq. (4.16). It follows that the complete contribution to $U_0^{(k)}$ is simply $(k-1)!/2$ times that given in Eq. (4.18),

$$|U_0^{(k)}| \geq \frac{2}{\pi R}\left(\frac{G_F a_n}{8\pi\sqrt{2}R^2}\right)^k \frac{k!(k-1)!}{2k^{k+1}} = \frac{1}{\pi R}\left(\frac{G_F a_n}{8\pi\sqrt{2}R^2}\right)^k \frac{(k!)^2}{k^{k+2}}. \tag{4.19}$$

It is instructive to consider the expression for $U_0^{(4)}$ in greater detail to understand the origin of the factor $(k-1)!/2$. For brevity define

$$u = r_{12} \qquad v = r_{23} \qquad w = r_{34} \qquad x = r_{41} \qquad y = r_{24} \qquad z = r_{13}. \tag{4.20}$$

From Eq. (4.16) the complete expression for $U_0^{(4)}$ is then given by

$$U_0^{(4)} = \frac{4}{\pi}\left(\frac{G_F a_n}{2\pi\sqrt{2}}\right)^4 4! \int_0^{2R} du\, dv\, dw\, dx\, dy\, dz\, \mathcal{P}^{(4)}(u,v,w,x,y,z)$$
$$\times \left[\frac{1}{uvwx(u+v+w+x)^5} + \frac{1}{vxyz(v+x+y+z)^5} + \frac{1}{uwyz(u+w+y+z)^5}\right], \tag{4.21}$$

where the three terms in square brackets arise from diagrams (a), (b), and (c) respectively in Fig. 2. We note that $\mathcal{P}^{(4)}(u,v,w,x,y,z)$ is necessarily a symmetric function of its arguments, and that the expression in square brackets is also a symmetric function of the same arguments. It then follows that whatever the explicit functional form of $\mathcal{P}^{(4)}(u,v,w,x,y,z)$, each of the 3 terms in square brackets contributes equally to the integral. For the k-body contribution the expression in square brackets would contain $(k-1)!/2$ terms, which is the origin of the factor $k!/2k$ in Eq. (4.19).

As we have noted previously the many body contributions are dominated by terms with $k \simeq N$, and for these terms the bound in Eq. (4.19) is useful. However, if we wish to obtain a closed-form expression for the sum, then the bound in Eq. (4.19) cannot be taken over directly, since the individual terms in the series alternate in sign, as can be seen from Eq. (4.16). We thus wish to replace the bound by a more precise estimate of the integral, and this can be done using the "mean value approximation," as discussed in Appendix C. From the preceding discussion, $U_0^{(k)}$ can be obtained by multiplying the expression in Eq. (C15) by $k!/2k$ which gives, for even k,

$$U_0^{(k)} \simeq \frac{2i^k}{\pi R}\left(\frac{G_F a_n}{2\pi\sqrt{2}R^2}\right)^k \frac{(k!)^2}{k^{k+2}}. \tag{4.22}$$

Eq. (4.22) is the starting point of our discussion in the next section of the combinatoric factors arising in a system of N particles.

V. COMBINATORICS FOR MANY-BODY SYSTEMS AND NUMERICAL RESULTS

A. Combinatorics

The expression for $U_0^{(k)}$ in Eq. (4.22) represents the k-body contribution to the neutrino-exchange energy in the approximation of retaining only the 0-derivative terms. As we noted in Sec. II, for a system of N particles there are $\binom{N}{k}$ such (identical) terms, and hence the total neutrino-exchange energy for a spherical distribution of N particles is given by

$$\begin{aligned} W &\simeq U^{(2)}\binom{N}{2} + \sum_{\substack{k=4\\ even}}^{N} U_0^{(k)}\binom{N}{k} \\ &= W^{(2)} + \sum_{\substack{k=4\\ even}}^{N} \frac{2i^k}{\pi R}\left(\frac{G_F a_n}{2\pi\sqrt{2}R^2}\right)^k \frac{(k!)^2}{k^{k+2}}\binom{N}{k}. \end{aligned} \tag{5.1}$$

$U^{(2)}$ is the 2-body contribution and, since it can be evaluated exactly, the full expression in Eq. (4.2) will be used. When calculating the neutrino-exchange energy in a white dwarf or a neutron star, $W^{(2)} = U^{(2)}\binom{N}{2}$ is completely negligible and hence it can be dropped from the sum over k when convenient.

The sum over k in Eq. (5.1) can be evaluated in closed form by making use of the Stirling approximation for $k!$, which is valid for large k:

$$k! \simeq \sqrt{2\pi}k^{k+1/2}e^{-k}. \tag{5.2}$$

Combining Eqs. (5.1) and (5.2) we have

$$W \simeq W^{(2)} + \sqrt{\frac{2}{\pi}}\,\frac{2}{R}\sum_{\substack{k=4\\ even}}^{N} \frac{i^k k!}{k^{3/2}}\left(\frac{G_F a_n}{2\pi\sqrt{2}eR^2}\right)^k \binom{N}{k}, \tag{5.3}$$

where $\ln e = 1$. The remaining $k!$ will be replaced shortly by an integral representation which, when evaluated by the saddle-point method, amounts to a second application of the Stirling approximation. The sum over k can be further simplified by noting that were we to apply the Stirling approximation again at this stage we could write

$$\frac{k!}{k^{3/2}} \simeq \sqrt{2\pi}\frac{k^{k+1/2}}{k^{3/2}}e^{-k} = \sqrt{2\pi}\exp[(k+1/2)\ln k - k - (3/2)\ln k]. \tag{5.4}$$

Since we are interested in evaluating the sum for $k \leq N = \mathcal{O}(10^{57})$, the term $(3/2)\ln k$ could be dropped relative to $k\ln k$ in Eq. (5.4), and hence the factor $k^{3/2}$ in Eq. (5.3) could also be dropped. However, a better approximation is to simply replace $k^{3/2}$ by $N^{3/2}$, noting that W is dominated by terms with $k \simeq N$. (This approximation also gives the smallest estimate for W.) Although the factor $k^{3/2}$ is negligible from a quantitative point of view, it can (with some effort) be reinstated if necessary, as we demonstrate in Appendix D. In the approximation of replacing $k^{3/2}$ by $N^{3/2}$, W becomes

$$W \simeq W^{(2)} + \sqrt{\frac{2}{\pi}}\,\frac{2}{R}\,\frac{1}{N^{3/2}} \sum_{\substack{k=4 \\ even}}^{N} k!(i\Gamma_R)^k \binom{N}{k}, \tag{5.5}$$

where

$$\Gamma_R = \frac{G_F a_n}{2\pi\sqrt{2}eR^2}. \tag{5.6}$$

Although the lower limit on the sum in Eq. (5.5) is $k = 4$, the summation can be extended down to $k = 0$ by subtracting the $k = 0$ and $k = 2$ contributions at the end. We thus consider the sums,

$$\sum{}^{(e)} = \sum_{\substack{k=0 \\ even}}^{N} k!(i\Gamma_R)^k \binom{N}{k}, \tag{5.7}$$

$$\sum{}^{(\pm)} = \sum_{k=0}^{N} k!(\pm i\Gamma_R)^k \binom{N}{k}, \tag{5.8}$$

where in $\sum^{(\pm)}$ the summation extends over all k. The quantity we want is $\sum^{(e)}$, and from Eqs. (5.7) and (5.8) it is given by

$$\sum^{(e)} = \frac{1}{2}\left[\sum^{(+)} + \sum^{(-)}\right]. \tag{5.9}$$

To evaluate $\sum^{(\pm)}$ we introduce into Eq. (5.8) the familiar integral representation for $k!$,

$$k! = \int_0^\infty du\, e^{-u} u^k, \tag{5.10}$$

which gives

$$\sum^{(\pm)} = \int_0^\infty du\, e^{-u} \sum_{k=0}^{N} (\pm iu\Gamma_R)^k \binom{N}{k}. \tag{5.11}$$

The sum of the binomial series in Eq. (5.11) can be expressed in closed form [37],

$$\sum_{k=0}^{N} x^k a^{N-k} \binom{N}{k} = (a+x)^N, \tag{5.12}$$

where $a = 1$ in the present case. Combining Eqs. (5.11) and (5.12) gives

$$\sum^{(\pm)} = \int_0^\infty du\, e^{-u}\,(1 \pm iu\Gamma_R)^N. \tag{5.13}$$

The integral in Eq. (5.13) can be readily evaluated by the saddle-point method [38]. We introduce the variable t defined by

$$1 \pm iu\Gamma_R = \pm iNt, \tag{5.14}$$

where the upper (lower) sign applies to $\sum^{(+)}$ ($\sum^{(-)}$). $\sum^{(+)}$ can then be written in the form

$$\sum^{(+)} = g(N) \int_{t_{min}}^{t_{max}} dt\, e^{Nf(t)}, \tag{5.15}$$

where

$$g(N) = \frac{N}{\Gamma_R} \exp\left[-\frac{i}{\Gamma_R} + N\ln(iN)\right], \tag{5.16}$$

$$f(t) = \left(-\frac{t}{\Gamma_R} + \ln t\right), \tag{5.17}$$

and t_{min} (t_{max}) corresponds to $u = 0$ ($u = \infty$) in Eq. (5.14). Using Ref. [38], $\sum^{(+)}$ is then given by

$$\sum\nolimits^{(+)} \simeq \frac{\sqrt{2\pi} g(N) e^{N f(t_0)}}{|N f''(t_0)|^{1/2}}, \tag{5.18}$$

where $t_0 = \Gamma_R$ is the saddle point, and the primes indicate differentiation. Combining Eqs. (5.16)–(5.18), we find

$$\begin{aligned}\sum\nolimits^{(+)} &\simeq \sqrt{2\pi N} \exp\left\{N\left[\ln\left(\Gamma_R N\right) - 1\right] + i(N\pi/2 - 1/\Gamma_R)\right\} \\ &= \sqrt{2\pi N}(\Gamma_R N/e)^N e^{i(N\pi/2 - 1/\Gamma_R)}.\end{aligned} \tag{5.19}$$

Since $\sum^{(-)} = (\sum^{(+)})^*$, as can be seen from Eq. (5.11), $\sum^{(e)}$ in Eqs. (5.7) and (5.9) is given by

$$\sum\nolimits^{(e)} \simeq (-1)^{N/2}\sqrt{2\pi N}(\Gamma_R N/e)^N \cos(1/\Gamma_R). \tag{5.20}$$

Noting that only even values of N contribute to $\sum_k$, it follows that the factor $(-1)^{N/2}$ is always real and alternates in sign. We also note that because the expression on the right hand side of Eq. (5.18) is the origin of both the Stirling approximation for $k!$ and the result for $\sum^{(e)}$ above, it follows that both factors of $k!$ in Eq. (5.1) have been treated in a consistent way. Combining Eqs. (5.5), (5.7), and (5.20) the neutrino-exchange energy W can be written in the form

$$W \simeq W^{(2)} + \sqrt{\frac{2}{\pi}}\,\frac{2}{R}\,\frac{1}{N^{3/2}}\left(\sum\nolimits^{(e)} - \sum\nolimits^{(0)} - \sum\nolimits^{(2)}\right), \tag{5.21}$$

where $\sum^{(0)}$ and $\sum^{(2)}$ are the $k = 0$ and $k = 2$ contributions to the sum over k in Eq. (5.7) which must be subtracted out. From Eq. (5.7),

$$\sum\nolimits^{(0)} + \sum\nolimits^{(2)} = 1 - N(N-1)\Gamma_R^2, \tag{5.22}$$

and hence

$$\begin{aligned}W \simeq{}& (-1)^{N/2}\frac{4}{RN}\left(\frac{\Gamma_R N}{e}\right)^N \cos(1/\Gamma_R) \\ &+ \left[W^{(2)} - \sqrt{\frac{2}{\pi N^3}}\left(\frac{2}{R}\right) + \frac{N(N-1)}{2e^2}\sqrt{\frac{1}{2\pi^5 N^3}}\left(\frac{G_F^2}{R^5}\right)\right].\end{aligned} \tag{5.23}$$

For purposes of setting a bound on the neutrino mass the expression in square brackets is negligible and can be dropped. Using Eq. (5.6) we can then write Eq. (5.23) as

$$W \simeq (-1)^{N/2} \frac{4}{RN} \left(\frac{G_F a_n N}{2\pi\sqrt{2} e^2 R^2} \right)^N \cos(1/\Gamma_R). \tag{5.24}$$

Although the expression in Eq. (5.24) is the result of summing over all $k \leq N$ in Eq. (5.5), it can be shown that this expression is effectively the contribution of a few terms which dominate the sum over k. One way of seeing this is to consider the ratio of successive terms in Eq. (5.5). Dropping the factor $k^{-3/2}$ as discussed above we find

$$\left| \frac{W^{(k+2)}}{W^{(k)}} \right| \simeq \Gamma_R^2 (N-k)(N-k-1). \tag{5.25}$$

As we demonstrate in Eq. (5.36) below, the product $\Gamma_R N$ is of order 10^{11}, and hence for small k the ratio in Eq. (5.25) is much larger than unity. It follows that successive terms in $\sum_k$ initially make increasingly larger contributions. As noted in Sec. II, however, the rate at which this ratio increases is itself a decreasing function for increasing k, and eventually a value k_{max} is reached for which the ratio is unity. From Eq. (5.25) k_{max} is given by

$$\Gamma_R^2 (N - k_{max})^2 \simeq 1, \tag{5.26}$$

$$k_{max} \simeq N - 1/\Gamma_R. \tag{5.27}$$

As noted in Sec. II, the fact that the individual terms in $\sum_k$ reach a maximum is a consequence of the behavior of $\binom{N}{k}$ for large k, and of Eq. (2.9) in particular. To calculate $W^{(max)}$, the value of $W^{(k)}$ corresponding to k_{max}, we can set $k_{max} \simeq N$ and $(1/\Gamma_R) \simeq 0$ since $N \gg 1/\Gamma_R$ (see below). From Eq. (5.5) we then have

$$W^{(max)} \simeq (-1)^{N/2} \frac{4}{RN} \left(\frac{\Gamma_R N}{e} \right)^N, \tag{5.28}$$

where the Stirling approximation for $N!$ has been used. Eq. (5.28) agrees with the result obtained previously in Eq. (5.24), up to the factor $\cos(1/\Gamma_R)$ which is the "memory" of the sum over k as we discuss below. It follows from Eq. (5.25) that for practical purposes one can approximate W reasonably well by the single term $W^{(max)}$.

B. Numerical Results

We proceed to evaluate W numerically for a typical neutron star which we take to be the observed pulsar in the Hulse-Taylor binary system PSR 1913+16 [39–41]. The mass M_1 of this pulsar is accurately known [40,41],

$$M_1 = 1.4411(7) M_\odot, \tag{5.29}$$

and hence the mass of a typical neutron star will be taken to be

$$M = 1.4 M_\odot = 2.8 \times 10^{33} \text{ g}. \tag{5.30}$$

To calculate the number of neutrons N we ignore the contribution to M from gravitational binding energy [see Eq. (5.43) below], and assume that the neutron star is composed exclusively of neutrons. Using Eq. (5.30) then leads to

$$N = 1.7 \times 10^{57}. \tag{5.31}$$

The radius R of the neutron star, although not directly observable, can be inferred in various models. We assume the nominal value $R = 10\text{ km} \equiv R_{10}$ which corresponds to a mass density ρ_m and a number density ρ given by

$$\rho_m = 6.7 \times 10^{14} \text{ g cm}^{-3}, \tag{5.32}$$

$$\rho = 4.0 \times 10^{38} \text{ cm}^{-3}. \tag{5.33}$$

In what follows we will assume these values of R and ρ, which are typical of the results that arise in existing models of neutrons stars [42,43]. Combining Eqs. (5.30)–(5.32) with $|a_n| = 1/2$ and reinstating $\hbar$ and c, we find

$$\frac{(G_F/\hbar c)N}{R_{10}^2} = 7.6 \times 10^{12}, \tag{5.34}$$

$$\Gamma_R \equiv \frac{(G_F/\hbar c)|a_n|}{2\pi\sqrt{2}eR_{10}^2} = 9.4 \times 10^{-47} = \frac{1}{1.1 \times 10^{46}}, \tag{5.35}$$

$$\Lambda_R \equiv \frac{(G_F/\hbar c)N|a_n|}{2\pi\sqrt{2}e^2R_{10}^2} = 1.2\times 10^{11}|a_n| = 5.8\times 10^{10}, \tag{5.36}$$

$$\begin{aligned} W &= (-1)^{N/2}\frac{4\hbar c}{R_{10}}\left(6.0\times 10^{-58}\right)\left(5.8\times 10^{10}\right)^{1.7\times 10^{57}}\cos(1/\Gamma_R) \\ &= 10^{(2\times 10^{58}-57-10)}(-1)^{N/2}\cos(1/\Gamma_R)\ \text{ eV}. \end{aligned} \tag{5.37}$$

In Eq. (5.37) the three terms in the exponent of 10 arise, respectively, from $(\Gamma_R N/e)^N$, $1/N$, and $4\hbar c/R_{10}$ (in eV). For the ratio W/Mc^2 we find

$$\frac{W}{Mc^2} = (-1)^{N/2}10^{(2\times 10^{58}-57-76)}\cos(1/\Gamma_R). \tag{5.38}$$

Leaving aside the cosine factor, to which we will return shortly, we see that the neutrino-exchange energy is significantly larger than the known mass-energy of the 1913+16 pulsar. For later purposes we note that in a neutron star $W^{(k)}/Mc^2$ exceeds unity for $k \geq 8$, which is a relatively low order perturbation. Using Eq. (5.1) we find that for the 0-derivative contribution,

$$W^{(8)} = \frac{2i^8\hbar c}{\pi R_{10}}\frac{(8!)^2}{8^{10}}\left[\frac{(G_F/\hbar c)|a_n|}{2\pi\sqrt{2}R_{10}^2}\right]^8\binom{N}{8} = 5.0\times 10^{77}\text{ eV}, \tag{5.39}$$

$$\frac{W^{(8)}}{Mc^2} = 3.2\times 10^{11}. \tag{5.40}$$

Another quantity of interest is the ratio of W to the total mass of the Universe M_U. We find:

$$M_U \simeq 4\times 10^{55}\text{ g}, \tag{5.41}$$

$$\frac{W}{M_U c^2} \simeq 10^{(2\times 10^{59}-57-99)}. \tag{5.42}$$

It follows from Eqs. (5.38) and (5.42) that the neutrino-exchange energy in a single neutron star, when calculated in the Standard Model, exceeds the total mass-energy of the Universe, and this may be termed the "neutrino-exchange energy-density catastrophe." In estimating

M_U, we assume the average density ρ_U of the Universe to be $\rho_U \simeq 1 \times 10^{-29}$ g cm^{-3}, which is the critical density corresponding to a Hubble constant of (80 ± 17) km s^{-1} Mpc^{-1} [44,45], and we have taken the radius of the Universe to be $\simeq 1 \times 10^{10}$ ly [45].

We return to discuss the sign of W in Eqs. (5.24) and (5.38). As noted previously, the factor $(-1)^{N/2}$ alternates in sign, and hence the sign of W depends on N, as well as on Γ_R through $\cos(1/\Gamma_R)$. Either sign of W leads to a catastrophic outcome: $W > 0$ would correspond to a large repulsive force against which the neutron star would be unstable, whereas $W < 0$ would cause the neutron star to collapse to a black hole. In either case neutron stars as we know them would not exist. In this context it is helpful to recall that the classical gravitational potential energy V_{GRAV} is approximately given by

$$V_{GRAV} \simeq -\frac{3}{5}\frac{G_N M^2}{R}, \tag{5.43}$$

where $G_N = 6.67259(85) \times 10^{-11}$ m^{3}kg^{-1}s^{-2} is the Newtonian gravitational constant. For the assumed values of M and N in Eqs. (5.30) and (5.31) we find

$$\frac{|V_{GRAV}|}{Mc^2} \simeq 0.124, \tag{5.44}$$

$$\frac{|V_{GRAV}|}{N} \simeq 117 \ \text{MeV/nucleon}. \tag{5.45}$$

It follows from Eqs. (5.38), (5.44), and (5.45) that the neutrino-exchange energy, whatever its sign, would dominate over V_{GRAV}, which is the largest contribution to the binding energy of a neutron star. We note in passing that since V_{GRAV} is negative, the effect of including gravitational binding energy in Eq. (5.30) would be to *increase* N in Eq. (5.31), which would make the neutrino-exchange energy density even larger than the value quoted in Eqs. (5.37) and (5.38).

We turn next to the factor $\cos(1/\Gamma_R) = \cos(2\pi\sqrt{2}eR^2/G_F a_n)$. This is similar to the oscillatory factors that arise in other many-fermion systems, an example being the Ruderman-Kittel interaction [46]. To better understand the significance of this factor it is instructive to ask what the expression for W would be if all terms in $\sum_k$ had the same sign (which we assume to be positive for illustrative purposes). The analog of Eq. (5.7) would then be

$$\widetilde{\sum}^{(e)} = \sum_{\substack{k=0\\ even}}^{N} k!\Gamma_R^k \binom{N}{k}, \tag{5.46}$$

and hence from Eqs. (5.7) and (5.46),

$$\widetilde{\sum}^{(e)} = \widetilde{\sum}^{(e)}(\Gamma_R) = \sum^{(e)}(\Gamma_R \to -i\Gamma_R). \tag{5.47}$$

It follows from Eq. (5.47) that W in Eq. (5.24) would be replaced by $\widetilde{W}$ where

$$\widetilde{W} \simeq \frac{4}{RN}\left(\frac{G_F a_n N}{2\pi\sqrt{2}e^2 R^2}\right)^N \cosh(1/\Gamma_R), \tag{5.48}$$

and hence the difference between W and $\widetilde{W}$ is the replacement

$$(-1)^{N/2}\cos(1/\Gamma_R) \to \cosh(1/\Gamma_R). \tag{5.49}$$

Since $\cosh(1/\Gamma_R)$ never vanishes, $\widetilde{W}$ is always nonzero for the values of the parameters we are assuming. By contrast $\cos(1/\Gamma_R)$ *can* be zero for some set of parameters. Hence W can in principle vanish, but only when $2/(\pi\Gamma_R)$ coincides with an odd integer to $\sim 10^{59}$ decimal places. Not only would this be unphysical for even a single neutron star, it could hardly be supposed that $\cos(1/\Gamma_R)$ would vanish in this way for each of the more than 500 known pulsars [47]. From the recent catalog of 558 pulsars by Taylor *et al.* [47], we note that there is a significant variation in the period P and spin-down rate $\dot{P}$ both of which depend on the internal structure of the pulsars. Absent an overarching (and presently unknown) symmetry principle which would force W to vanish, it is difficult to imagine that this could happen coincidentally in every case. This argument is further strengthened by noting that the catalog of Taylor *et al.* specifically excludes accretion-powered systems which are detected by their X-ray emissions. For these, the in-falling matter would continuously change the parameters of the pulsar at a level that would preclude the possibility that $\cos(1/\Gamma_R)$ would always be zero.

In the preceding discussion we have considered the effects of the neutrino-exchange energy in an idealized non-rotating neutron star. Including the effects of rotation would further bolster the preceding arguments, since the variation in P implies a variation in the internal

structure of the pulsars, as noted above. Proceeding further one could also ask whether a neutron-star could even come into existence in the presence of an energy density as large as would arise from neutrino-exchange. Although the answer to this question could lead to a more stringent limit on neutrino masses than is obtained from our "static" picture, such dynamical considerations are beyond the scope of the present paper. However, even in the absence of a detailed calculation one can argue that normal stars could not evolve into neutron stars in the presence of neutrino-exchange. This follows by noting that as soon as a star evolves to the stage where $(\Gamma_R N/e)$ in Eq. (5.23) exceeds unity, its mass-energy not only becomes unphysically large, but also alternates in sign as each pair of neutrons is accreted. Hence if at some stage $(-1)^{N/2}$ were -1, the addition of 2 neutrons would require overcoming an unphysically large repulsive energy barrier, which thus acts to prevent further accretion.

Having demonstrated that the oscillatory factor $\cos(1/\Gamma_R)$ cannot prevent the neutrino-exchange energy-density catastrophe, we study some of the implications which follow from the presence of this factor in the expression for W. To start with, the sign of the 0-derivative contribution to W in Eqs. (5.24) and (5.37) depends on the product $(-1)^{N/2}\cos(1/\Gamma_R)$. To determine the sign and magnitude of $\cos(1/\Gamma_R)$, Γ_R itself would have to be known to an unphysically large number of decimal places. (For illustrative purposes, if Γ_R in Eq. (5.35) were *exact*, then $\cos(1/\Gamma_R) = +0.613$ [48].) Hence as a practical matter, the overall sign of W for a given neutron star is difficult to determine, but is also relatively unimportant given that either sign leads to a difficulty. There is, however, an interesting implication of the variation in sign of W as a function of N and Γ_R: Rather than limit our attention to the parameters of the $\sim$ 500–600 known neutron stars that are pulsars, we can view any spherical volume of radius r inside a neutron star as a collection of $N(r)$ neutrons to which the preceding arguments apply. In order to have a stable static neutron star all such subvolumes must be in equilibrium, notwithstanding the variation of the sign and magnitude of $W(r)$ with r. By pursuing this argument, we are led in Section VIII to the bound on m quoted in Eq. (8.12) below.

We conclude the present discussion by presenting a physical interpretation of the factor $(-1)^{N/2}\cos(1/\Gamma_R)$. Combining Eqs. (5.28) and (5.48) we can write

$$W \simeq W^{(max)}\cos(1/\Gamma_R), \tag{5.50}$$

$$\widetilde{W} \simeq (-1)^{N/2} W^{(max)}\cosh(1/\Gamma_R). \tag{5.51}$$

We see from Eqs. (5.50) and (5.51) that the overall scale of both W and $\widetilde{W}$ is primarily determined by the single term $W^{(max)}$ in Eq. (5.28). The factor $\cos(1/\Gamma_R)$ in Eq. (5.50), which acts to suppress W, can then be understood as the "memory" of the cancellations arising from the remaining terms in $\sum^{(e)}$. By way of contrast the factor $\cosh(1/\Gamma_R)$ in Eq. (5.51) enhances the contribution from $W^{(max)}$, and thus reflects the effect of coherently adding all the same-sign contributions in $\widetilde{\sum}^{(e)}$. For practical purposes the magnitude of the neutrino-exchange energy-density is thus determined by $W^{(max)}$, for which the bound in Eq. (4.19) can be used.

While on the subject of cancellations it is worthwhile to consider another possible source of cancellations, namely the effects of the Pauli exclusion principle for neutrinos. The Pauli effect for neutrinos can enter in two ways: Firstly, the exchanged $\nu\bar{\nu}$ pairs can interfere with one another and, secondly, the exchanged neutrinos could interfere with a possible neutrino sea produced by the neutron star or white dwarf. With respect to the first possibility, the exchanged neutrinos do in fact interfere with one another, but these effects are rigorously taken into account from the outset in the Schwinger-Hartle formalism, and at all stages in the present derivation. Since the Schwinger formula in Eq. (B34) is simply a convenient way of summing all the one-loop Feynman amplitudes (which clearly incorporate the Pauli exclusion principle), all such effects are automatically included. For example, the factor of (-1) from the closed neutrino loop, which is a consequence of the anticommutativity of the neutrino field operators, is already built in. Moreover, the expression for W in Eq. (5.24) is the result of coherently summing the k-body contributions, with all the relevant phases determined by the Pauli principle. Hence all coherent effects of the Pauli exclusion principle for the exchanged neutrinos are accounted for in the present results.

Consider next the possibility that neutrino-exchange is somehow suppressed by the presence of a sea of physical neutrinos in a neutron star. It is easy to demonstrate that such a suppression cannot take place, because the density of physical neutrinos in a neutron star is essentially zero [49]. From Ref. [49] we note that the mean free path λ_n for the scattering of ν_e from neutrons is given by

$$\lambda_n \simeq 300\,\mathrm{km}\,\frac{\rho_{nuc}}{\rho_m}\left(\frac{100\,\mathrm{keV}}{E_\nu}\right)^2 . \tag{5.52}$$

Here ρ_{nuc} is the nuclear density, which is taken to be $\rho_{nuc} = 2.8 \times 10^{14}$ g cm^{-3}, and ρ_m is the neutron star density. For $E_\nu = 10$ eV, the energy scale set by $G_F \rho_n$, we find $\lambda_n \simeq 1 \times 10^{10}$ km, which means that a neutron star is transparent to low energy neutrinos [49]. Since low-energy neutrinos cannot be trapped inside a neutron star, there is no possibility of a neutrino sea in a neutron star. The few neutrinos that are present (in transit) at any instant are obviously incapable of reducing the effective Fermi constant by a factor of order 10^{11}, which is what would be needed to resolve the energy-density problem. Finally we recall from Table II that neutrino-exchange leads to a large energy-density in white dwarfs as well as in neutron stars. Hence this difficulty cannot be resolved by invoking a physical neutrino sea, since the mean free path of a neutrino in a white dwarf would be even larger than in a neutron star, so even fewer neutrinos could be trapped to form a neutrino sea.

Returning to Eqs. (5.24) and (5.28) we note that these results arise from the 0-derivative contribution $U_0^{(k)}$ in Eq. (5.1). We consider next the contributions to W from the derivative terms in Eqs. (3.55)–(3.58). Since these terms introduce no new dimensional factors, their contribution to W must have the same dependence on $(G_F a_n/R^2)^k$ as in Eq. (5.1), although the k-dependent coefficients will in general be different. We consider three possibilities: a) The net contribution from the derivative terms is *smaller* than that from the 0-derivative terms. In this case Eq. (5.1) is an adequate approximation to W, and the previous conclusions apply as before. b) The net contribution from the derivative terms is *larger* than that from the 0-derivative terms, a possibility suggested by the 4-body results. In this case the neutrino-exchange energy-density catastrophe poses an even greater difficulty than before,

and resolving the problems raised by the 0-derivative contribution is a necessary first step. It is evident that whatever k-dependent coefficient replaces $(k!)^2/(k^{k+2})$ in Eq. (5.1), the k-body contribution arising from the derivative terms will necessarily be proportional to the same factor

$$\left(\frac{(G_F/\hbar c)a_n}{2\pi\sqrt{2}R^2}\right)^k \binom{N}{k}, \tag{5.53}$$

that appears in the 0-derivative contribution. Hence whatever mechanism resolves the neutrino-exchange energy-density catastrophe for the 0-derivative terms, will also work for the derivative terms, as we discuss in greater detail in Sec. VIII below. c) The last possibility is that the derivative terms cancel the 0-derivative contribution so as to reduce $|W/Mc^2|$ in Eq. (5.38) to a number of order unity (or more realistically to ~ 0.1). Such a cancellation would have to occur in each of the 500–600 known pulsars, and (as we discuss below) in each of $\sim$2000 known white dwarfs. Since the derivative terms have a different dependence on $\vec{r}_{ij}$ than the 0-derivative terms, they would also depend differently on the matter distribution in the neutron star, were we to allow for a varying density in a more realistic calculation. Thus, to avoid the neutrino-exchange energy-density problem would require at a minimum an almost exact cancellation between the derivative and 0-derivative terms in each pulsar, notwithstanding the fact that the former depend differently from the latter on the matter distribution, which itself varies from one pulsar to another. As noted previously, such cancellations would be unphysical absent some (presently unknown) symmetry principle. We are thus led to the conclusion that the derivative terms can exacerbate the energy-density problem arising from the 0-derivative terms, but they cannot resolve it in the framework of the present (Standard Model) calculation.

As noted in the preceding paragraph, neutrino-exchange leads to an energy-density catastrophe in white dwarfs as well as in neutron stars. Although neutron stars provide the most stringent lower bound on m, the weaker bound from white dwarfs is nevertheless important because its existence makes it less likely that this problem can be resolved without introducing massive neutrinos. For example, the fact that white dwarfs contain roughly com-

parable numbers of electrons, protons, and neutrons, rules out any attempt to address the problem in neutron stars by a mechanism which somehow suppresses the neutrino-neutron coupling. A 1987 catalog by McCook and Scion [50] lists 1279 white dwarfs which have been identified spectroscopically, and the current catalog contains more than 2000 entries. To demonstrate that the energy-density arising from neutrino-exchange in white dwarfs is also unphysically large, we consider two representative white dwarfs whose masses and radii are reasonably well-known [51]. The relevant parameters for these two white dwarfs, Sirius B and 40 Eri B, are summarized in Table II. Following Ref. [51] we assume that the interiors of these white dwarfs are predominantly carbon and oxygen, in which case these stars contain approximately equal numbers of neutrons, protons, and electrons as noted previously. The combinatorics of the many-body diagrams involving three types of particles are somewhat complicated, but for present purposes it is sufficient to evaluate the purely electronic contribution, noting from Appendix A that electrons have the strongest coupling to neutrinos. The previous formalism for neutron stars can then be taken over immediately and leads to the results given in the last two lines of Table II where,

$$\Lambda_R = \frac{(G_F/\hbar c) N_e a_e}{2\pi\sqrt{2} e^2 R^2}. \tag{5.54}$$

Here N_e is the total number of electrons, $a_e = (2\sin^2\theta_W + 1/2) \simeq 0.964$ is the e-ν coupling constant [see Eq. (A9c)], and we assumed for illustrative purposes that the electron density is spatially constant. It follows from Table II that for both Sirius B and 40 Eri B the energy arising from neutrino-exchange exceeds the mass-energy of the Universe, just as in the case of neutron stars.

The results in Table II raise the possibility that the existence of a large neutrino-exchange energy-density may be not be limited to neutron stars and white dwarfs. From Eq. (5.24) we see that for a single particle species $j = n, e,$ or p, this happens when

$$\Lambda_R = \frac{(G_F/\hbar c) N_j a_j}{2\pi\sqrt{2} e^2 R^2} > 1, \tag{5.55}$$

where a_j is the coupling to neutrinos. For the Sun the contribution to Λ_R from electrons alone slightly exceeds unity, which suggests that the very existence of the Sun, and by

extension life on Earth, is evidence for the necessity of massive neutrinos. Whether or not the Sun itself proves to be unstable against neutrino-exchange can only be determined by more detailed calculations. However, the possibility that other stars exist for which this is true indicates that the problems arising from neutrino-exchange can only be resolved by a universal mechanism that would apply to large numbers of rather different celestial objects.

VI. ALTERNATIVES TO MASSIVE NEUTRINOS

We consider in this section several alternatives to the introduction of massive neutrinos as a means for resolving the neutrino-exchange energy-density problem. It will be argued that no presently known mechanism except for massive neutrinos is compatible with existing experimental and theoretical constraints.

A. Deviations of the Neutrino Couplings from the Standard Model Predictions

As noted in the Introduction, the magnitude of W in Eq. (5.38) is determined by dimensional arguments and combinatoric considerations, and hence is insensitive to the detailed form of the neutrino coupling. To date there are no known discrepancies between the predictions of the Standard Model and experiment (see Appendix A), and the agreement is at the level of a few percent or better [6]. It would thus be difficult to understand how the effective Fermi constant $G_F|a_n|$ could differ from the Standard Model value by a factor of order 10^{11}, which is what would be required to avert a large energy-density. Similar considerations apply to the possibility that the neutral current interaction contains small admixtures of S, P, and T in addition to the usual V, A currents.

In this connection we recall that the mass density ρ_m in Eq. (5.33), $\rho_m = 6.7 \times 10^{14}$ g cm^{-3}, is comparable to nuclear density, $\rho_{nuc} = 2.8 \times 10^{14}$ g cm^{-3}. It follows that the success of conventional theory in explaining weak interactions in nuclei [52] strongly suggests that the weak neutrino-hadron and neutrino-electron couplings in neutron stars do not differ significantly from the predicted Standard Model values. This is supported by the general

agreement between theory and observation for such processes as supernova formation [53]. We further observe that even smaller renormalization effects are expected in white dwarfs, where a large neutrino-exchange energy-density nonetheless exists. Finally, we note that various astrophysical arguments support both the spatial and temporal constancy of G_F [54]. Taken together these arguments make it unlikely that the present difficulties can be resolved by modifying the neutrino coupling constants.

B. Cancellations Among ν_e, ν_μ, and ν_τ

The expression for W in Eq. (5.24) represents the contribution from a single neutrino species, $\nu_e\bar{\nu}_e$ for example. It might be argued perhaps the contributions from $\nu_e\bar{\nu}_e$, $\nu_\mu\bar{\nu}_\mu$, and $\nu_\tau\bar{\nu}_\tau$ could cancel amongst themselves in such a way as to reduce W to a physically reasonable value. However, such a scenario is in conflict with e-μ-τ universality, which implies that the couplings of ν_e, ν_μ, and ν_τ to n, p, or e have the same signs as well as the same magnitudes, and hence cannot produce the necessary cancellation. Experimental support for e-μ-τ universality comes from a number of sources including the equality of the partial decay rates $\Gamma(Z^0 \rightarrow \ell^+\ell^-)$ where $\ell = e, \mu, \tau$ [6].

C. Breakdown of Perturbation Theory

It might be suggested that the fact that W is unphysically large indicates that our use of the Schwinger formula in Eq. (B34) is not valid for some reason. This is a possibility that cannot be completely excluded at present, particularly in light of recent interest in the breakdown of perturbation theory in high orders [55]. We can argue, however, that this is not likely to lead to a resolution of the energy-density difficulties for reasons we now discuss. To start with, the expression for W in Eq. (5.24) is the sum of a finite number of contributions, each of which is well-behaved. Moreover, for $k \geq 6$ the one-loop k-body diagrams are finite, even in the absence of any regularization scheme. It follows that the magnitude of W cannot be explained as an artifact of the manner in which the potentials are extracted from

one-loop amplitudes. Finally, the actual k-body neutrino-exchange potentials which arise from the Schwinger-Hartle formalism are themselves well-behaved; it is only when particular values of the parameters G_F, N, R, ρ, and m are used (e.g., $m = 0$) that unphysical results emerge. This suggests that the resolution of the present difficulties lies in establishing what the correct value of m is rather than in finding a possible breakdown of perturbation theory. The previously discussed calculation of the Coulomb energy of a nucleus provides a useful analogy. Suppose that it was believed, for whatever reason, that for a nucleus with A nucleons the nuclear radius R in Eq. (2.6) was given by $R = A^{1/3} r_0$, with $r_0 = 0.01$ fm. The resulting Coulomb repulsion would be large enough to destabilize all nuclei, so matter as we know it could not exist. Although one could attempt to resolve this "Coulomb catastrophe" by examining higher-order effects in perturbation theory, a more reasonable approach would be to first ask whether another value of the parameter r_0, say $r_0 \sim 1$ fm, was compatible with experiment. Analogously, we demonstrate in Sec. VIII below that if $m \gtrsim 0.4$ eV$/c^2$, then neutrino-exchange no longer leads to an energy-density catastrophe. This path has clear observational implications, and should these be shown to be incompatible with experiment, then a re-examination of perturbation theory would be appropriate along the lines we now consider.

As we have already noted, the final expression for W in Eq. (5.24) is the result of summing the contributions from a finite number of many-body potentials, each of which is well-behaved. It follows that if there is indeed a breakdown of perturbation theory, it cannot be due simply to a failure of the perturbation series to converge. One possibility which has been discussed recently [55] is the rapid growth of the cross sections for producing large numbers of scalar particles. It has been conjectured [55] that the combinatoric factor $n!$ which appears in the amplitude for producing n particles in the final state could overcome the smallness of the coupling constant in weakly coupled theories, and thus produce observably large effects at high energies. This has led to an examination of the need for unitarizing the tree-level amplitudes from which these effects arise, and more generally, to a discussion of whether such effects can in fact be seen.

Although the origin of the combinatoric factor $n!$ which is responsible for the growth of the tree-level contributions is similar to that for the factor $(k-1)!/2$ in Eq. (4.19), the neutrino-exchange problems depend critically on the presence of the factor $N!$ in $\binom{N}{k}$, which has no analog in the work described in Ref. [55]. Moreover, we have seen in Eq. (5.40) that the large neutrino-exchange energy-density arises in $\mathcal{O}(G_F^8)$, which is a relatively low order of perturbation theory compared to the effects considered in Ref. [55]. Finally, the self-energy of a neutron star arising from neutrino-exchange is a low-energy phenomenon, in contrast to the production processes in Ref. [55]. Hence the unitarization mechanism considered in that context would not be relevant here. For all these reasons it appears unlikely that a breakdown of perturbation theory along the lines contemplated in Ref. [55] could resolve the neutrino-exchange energy-density catastrophe in a neutron star or white dwarf. Nonetheless this remains an interesting avenue for future exploration. Since at present the only *known* viable mechanism is the assumption that neutrinos are massive, we explore the implications of $m \neq 0$ in the following sections.

VII. MANY-BODY INTERACTIONS FROM THE EXCHANGE OF MASSIVE NEUTRINOS

In this section we apply the formalism for massive neutrinos developed in Appendix E to the problem of calculating the many-body neutrino-exchange potentials. Our objective is to demonstrate that when $m \neq 0$, the neutrino-exchange energy can be approximated by replacing

$$1/R^2 \to e^{-mR}/R^2, \tag{7.1}$$

in Eq. (5.24). As noted in Appendix E this leads to "saturation" of the neutrino-exchange interaction, and ultimately to a bound on m as we describe below.

When $m \neq 0$ the couplings of neutrinos to other matter may be different from the predictions of the Standard Model. Although the details of whatever Hamiltonian would

emerge cannot be fully discerned at the present time, it is sufficient for present purposes to consider a simple phenomenological model in which ν_e, ν_μ, and ν_τ remain as the eigenstates of the Hamiltonian, with each having a nonzero mass. The modification of the neutrino couplings to matter can then be parameterized as in Eq. (7.2) below.

A. The 2-Body Potential when $m \neq 0$

The expression for $W^{(2)}$ when $m \neq 0$ can be obtained from the $m = 0$ expression in Eq. (3.3) by replacing $S_F^{(0)}(\vec{r}_{12}, E)$ with the expression given in Eq. (E18). In addition we allow for the possibility that the neutrino coupling may not be pure V-A when $m \neq 0$ by substituting

$$(1+\gamma_5) \rightarrow (1+b\gamma_5), \tag{7.2}$$

where b is a constant (which may be different for ν_e, ν_μ, and ν_τ). If there is also a modification of the overall strength of neutrino coupling, this can be accommodated by appropriately redefining G_F. After calculating the trace of the Dirac γ-matrices, the analog of Eq. (3.10) can be written in the form

$$W^{(2)} = \frac{2i}{\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^2 \int \rho_1\, d^3x_1 \int \rho_2\, d^3x_2 \int_{-\infty}^{\infty} dE \left\{(1+b^2)E^2 - [(1+b^2)\vec{\partial}_{12}\cdot\vec{\partial}_{21} - (1-b^2)m^2]\right\}$$
$$\times \left(\frac{i}{4\pi}\right)^2 \frac{1}{r_{12}r_{21}} \exp[i(r_{12}+r_{21})\sqrt{E^2-m^2}]. \tag{7.3}$$

Following the discussion in Sec. III, and using Eq. (E20), the 2-body potential $V^{(2)}$ is given by

$$V^{(2)} = -i\frac{G_F^2 a_n^2}{16\pi^3}\left\{(1+b^2)F_2(z) - [(1+b^2)\vec{\partial}_{12}\cdot\vec{\partial}_{21} - (1-b^2)m^2]F_0(z)\right\}\frac{1}{r_{12}r_{21}}, \tag{7.4}$$

where $z \equiv r_{12} + r_{21}$. The expression in curly brackets in Eq. (7.4) can be simplified by using Eq. (3.17) and the recurrence relation in Eq. (E30) which leads to

$$V^{(2)} = i\frac{G_F^2 a_n^2}{16\pi^3}\left\{(1+b^2)\frac{F_0^{(2)}(z)}{r_{12}r_{21}} - (1+b^2)\frac{\partial}{\partial r_{12}}\frac{\partial}{\partial r_{21}}\left[\frac{F_0(z)}{r_{12}r_{21}}\right] - 2m^2\frac{F_0(z)}{r_{12}r_{21}}\right\}. \tag{7.5}$$

After the derivatives in Eq. (7.5) are explicitly evaluated, and use is made of Eq. (E33), we find

$$V^{(2)} = -\frac{G_F^2 a_n^2}{4\pi^3}\left\{(1+b^2)\left[\frac{m^2}{r^3}K_1^{(1)}(2mr) - \frac{m}{2r^4}K_1(2mr)\right] - \frac{m^3}{r^2}K_1(2mr)\right\}, \tag{7.6}$$

where $r = r_{12} = r_{21}$. Note that the contributions proportional to $K_1^{(2)}$, which arise from the derivatives and from $F_0^{(2)}$, cancel against each other. Using Eq. (E35) $K_1^{(1)}(2mr)$ can be eliminated in favor of $K_0(2mr)$ and $K_1(2mr)$, and this leads to the final expression for $V^{(2)}$ when $m \neq 0$:

$$V^{(2)} = \frac{G_F^2 a_n^2}{4\pi^3}\left\{(1+b^2)\left[\frac{m}{r^4}K_1(2mr) + \frac{m^2}{r^3}K_0(2mr)\right] + \frac{m^3}{r^2}K_1(2mr)\right\}, \tag{7.7}$$

The result in Eq. (7.7), which is exact, can be compared to the corresponding result in Eq. (3.19) for the massless case by expanding $K_1(x)$ and $K_0(x)$ for $x \simeq 0$:

$$K_0(x) \simeq -\ln x, \tag{7.8}$$

$$K_1(x) \simeq 1/x. \tag{7.9}$$

The only contribution which survives as $m \to 0$ is from the term proportional to $1/r^4$, and this gives

$$V^{(2)}(r) \simeq \frac{G_F^2 a_n^2}{8\pi^3}\,\frac{(1+b^2)}{r^5}. \tag{7.10}$$

When $b = 1$, which is the value appropriate to the massless case, Eq. (7.10) reproduces Eq. (3.19) as expected. As noted in Appendix E, however, the regime of interest here is when $2mr \gg 1$, in which case the asymptotic expression in Eq. (E27) for $K_\nu(2mr)$ leads to the anticipated exponential fall-off of $V^{(2)}(r)$.

B. Absence of Odd-k Contributions when $m \neq 0$

In this subsection we demonstrate that for $m \neq 0$ it remains the case that there are no contributions to W for odd k. Using Eq. (3.29a) we have

$$C^{-1}(\gamma \cdot \eta - m)C = (-\gamma \cdot \eta - m)^T, \tag{7.11}$$

from which it follows that

$$C^{-1}S_{Fm}^{(0)}(\vec{r}_{ij}, E)C = S_{Fm}^{(0)T}(-\vec{r}_{ij}, -E) = S_{Fm}^{(0)T}(\vec{r}_{ji}, -E). \tag{7.12}$$

This is the same relation that holds in the massless case [see Eq. (3.30)], and because the sign of E is immaterial for the reasons discussed previously, we can write as before,

$$C^{-1}S_{Fm}^{(0)}(\eta_{ij})C = S_{Fm}^{(0)T}(\eta_{ji}). \tag{7.13}$$

Under C the modified neutrino coupling transforms as

$$C^{-1}(1 + b\gamma_5)C = (1 + b\gamma_5)^T, \tag{7.14}$$

and since both the coupling and the neutrino propagator behave under C just as they do in the massless case, the previous conclusion that there are no odd-k contributions follows when $m \neq 0$ as well.

It is instructive to illustrate the preceding conclusion by considering the 5-body contribution as an example. The contribution from the standard diagram is given in the static limit by

$$\begin{aligned} W_a^{(5)} = \frac{-i}{2\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^5 \int_{-\infty}^{\infty} dE \, \mathrm{Tr}\Big\{&\gamma_\mu(1+b\gamma_5)S_{Fm}^{(0)}(51)\gamma_\sigma(1+b\gamma_5)S_{Fm}^{(0)}(45)\gamma_\rho(1+b\gamma_5)S_{Fm}^{(0)}(34) \\ &\times \gamma_\lambda(1+b\gamma_5)S_{Fm}^{(0)}(23)\gamma_\nu(1+b\gamma_5)S_{Fm}^{(0)}(12)\Big\} T_{\mu\sigma\rho\lambda\nu}(x_1, x_2, x_3, x_4, x_5). \end{aligned} \tag{7.15}$$

In the static limit the expression in curly brackets is given by

$$\begin{aligned} \mathrm{tr}\,[\mathrm{Eq.}\ (7.15)] &= \mathrm{tr}\Big[(1-b\gamma_5)\bar{S}_{Fm}^{(0)}(51)(1+b\gamma_5)S_{Fm}^{(0)}(45)(1-b\gamma_5)\bar{S}_{Fm}^{(0)}(34) \\ &\qquad \times (1+b\gamma_5)S_{Fm}^{(0)}(23)(1-b\gamma_5)\gamma_4 S_{Fm}^{(0)}(12)\Big], \end{aligned} \tag{7.16}$$

where $\bar{S}_{Fm}^{(0)}(51) \equiv \gamma_4 S_{Fm}^{(0)}(51)\gamma_4$. Combining Eqs (7.13)–(7.16), the trace in Eq. (7.16) can be re-expressed in the form

$$\begin{aligned} \mathrm{tr}\,[\mathrm{Eq.}\ (7.15)] &= -\mathrm{tr}\Big[(1-b\gamma_5)S_{Fm}^{(0)}(-12)\gamma_4(1-b\gamma_5)S_{Fm}^{(0)}(-23)(1+b\gamma_5)\bar{S}_{Fm}^{(0)}(-34) \\ &\qquad \times (1-b\gamma_5)S_{Fm}^{(0)}(-45)(1+b\gamma_5)\bar{S}_{Fm}^{(0)}(-51)\Big]. \end{aligned} \tag{7.17}$$

The contribution from the diagram with the reversed loop momentum can be written down in a similar manner, and the analog of the right-hand side Eq. (7.17) is

$$\mathrm{tr}\left[(1+b\gamma_5)S^{(0)}_{Fm}(21)\gamma_4(1+b\gamma_5)S^{(0)}_{Fm}(32)(1-b\gamma_5)\bar{S}^{(0)}_{Fm}(43)(1+b\gamma_5)S^{(0)}_{Fm}(54)(1-b\gamma_5)\bar{S}^{(0)}_{Fm}(15)\right]. \tag{7.18}$$

Since $S^{(0)}_{Fm}(-ij) = S^{(0)}_{Fm}(ji)$ it follows that when the contributions in Eqs. (7.17) and (7.18) are added only those terms containing odd powers of b survive. All such terms can be eventually reduced to the trace of a product of γ-matrices containing a single γ_5, and from Eq. (A3) it is seen that any such trace is always proportional to $\epsilon_{\mu\nu\lambda\rho}$. By the arguments given previously in the massless case, all terms proportional to $\epsilon_{\mu\nu\lambda\rho}$ average to zero when integrated over a spherically symmetric matter distribution. This leads to the conclusion that there is no 5-body contribution to W, and by extension no contribution for any odd k, even when $m \neq 0$.

C. The 4-Body Potential for $m \neq 0$

In this subsection we obtain the explicit form of the 4-body potential, which is then used to infer the dependence of the general k-body potential on the neutrino mass m. This result forms the basis for the bound on m that we derive in the following section.

As in the 2-body case we begin with Eq. (B34) and substitute $(1+b\gamma_5)$ for $(1+\gamma_5)$, and the massive neutrino propagator in Eq. (E18) for the massless one. The formalism of Sec. III can then be taken over directly and, after the Dirac traces are evaluated, the expression for the 0-derivative contribution from Fig. 2(a) to the 4-body potential is given by

$$\begin{aligned} V_0^{(4)}(\vec{r}_{12},\vec{r}_{23},\vec{r}_{34},\vec{r}_{41}) = -\frac{i}{2\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^4 8\int_{-\infty}^{\infty} dE\,\left[E^4 c_4(b) + m^2E^2c_2(b) + m^4c_0(b)\right] \\ \times\,\Delta_{Fm}(\vec{r}_{12},E)\Delta_{Fm}(\vec{r}_{23},E)\Delta_{Fm}(\vec{r}_{34},E)\Delta_{Fm}(\vec{r}_{41},E), \end{aligned} \tag{7.19}$$

where

$$\Delta_{Fm}(\vec{r}_{12},E) = \frac{i}{4\pi r_{12}}\exp\left(ir_{12}\sqrt{E^2-m^2}\right), \tag{7.20}$$

$$c_4(b) = (1+b^2)^2 + 4b^2 \quad ; \quad c_2(b) = 2(1-b^2)(3+b^2) \quad ; \quad c_0(b) = (1-b^2)^2. \tag{7.21}$$

The expression for $V_0^{(4)}$ given in Eqs. (7.19)—(7.21) is the sum of the contributions from the standard diagram with both senses of the neutrino loop momentum. This introduces a factor of 2 in Eq. (7.19), and combined with a factor of 4 from the Dirac trace, accounts for the factor of 8. As will be clear from the ensuing discussion, it is sufficient for present purposes to evaluate the 0-derivative contribution to $V^{(4)}$, from which the dependence of $V^{(k)}$ on m can be inferred. From Eqs. (7.19) and (7.20),

$$\Delta_{Fm}(\vec{r}_{12},E)\Delta_{Fm}(\vec{r}_{23},E)\Delta_{Fm}(\vec{r}_{34},E)\Delta_{Fm}(\vec{r}_{41},E) = \left(\frac{i}{4\pi}\right)^4 \frac{1}{r_{12}r_{23}r_{34}r_{41}} \times \exp\left[i\sqrt{E^2-m^2}(r_{12}+r_{23}+r_{34}+r_{41})\right], \tag{7.22}$$

which allows $V_0^{(4)}$ to be expressed in the form

$$V_0^{(4)} = -i\frac{4}{\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^4 \left(\frac{i}{4\pi}\right)^4 \frac{1}{r_{12}r_{23}r_{34}r_{41}} \left[c_4(b)F_4(z) + m^2 c_2(b)F_2(z) + m^4 c_0(b)F_0(z)\right]. \tag{7.23}$$

Here $z = (r_{12}+r_{23}+r_{34}+r_{41})$, and the functions $F_n(z)$ are defined in Eq. (E20). Combining Eq. (7.23) with Eqs. (E30) and (E32) then leads to

$$V_0^{(4)} = -i\frac{4}{\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^4 \left(\frac{i}{4\pi}\right)^4 \frac{1}{r_{12}r_{23}r_{34}r_{41}} \times \left\{c_4(b)F_0^{(4)}(z) - [2c_4(b)+c_2(b)]m^2F_0^{(2)}(z) + [c_4(b)+c_2(b)+c_0(b)]m^4F_0(z)\right\}. \tag{7.24}$$

It is worth noting that there are mass-dependent contributions even in the limit of a pure V-A coupling ($b=1$). This follows from the observation that although $c_2(1) = c_0(1) = 0$, each of the mass terms receives a contribution from $c_4(1) = 8$. Combining Eqs. (7.24) and (E30) the complete expression for $V_0^{(4)}$ can be written as

$$V_0^{(4)} = -i\frac{4}{\pi}\left(\frac{G_F a_n}{\sqrt{2}}\right)^4 \left(\frac{i}{4\pi}\right)^4 \frac{1}{r_{12}r_{23}r_{34}r_{41}} \times \left\{c_4(b)\,2i\left[\left(\frac{2m^4}{z}+\frac{12m^2}{z^3}\right)K_0(mz) + \left(m^5+\frac{7m^3}{z^2}+m\frac{4!}{z^4}\right)K_1(mz)\right]\right.$$

$$- \left[2c_4(b) + c_2(b)\right] 2i \left[\frac{m^4}{z} K_0(mz) + \left(m^5 + m^3 \frac{2!}{z^2}\right) K_1(mz)\right]$$
$$+ \left[c_4(b) + c_2(b) + c_0(b)\right] 2im^5 K_1(mz) \Big\} . \tag{7.25}$$

As in the 2-body case we wish to check $V_0^{(4)}$ in the limit $m \to 0$. Using Eqs. (7.8) and (7.9) we note that since $x^n \ln x \to 0$ as $x \to 0$ all the terms containing $K_0(mz)$ vanish in the $m = 0$ limit. Among the terms containing $K_1(mz)$ only the term proportional to $1/z^4$ survives, and when $b = 1$ this gives

$$V_0^{(4)} \xrightarrow{m=0} \frac{4}{\pi} \left(\frac{G_F a_n}{2\pi\sqrt{2}}\right)^4 \frac{4!}{r_{12} r_{23} r_{34} r_{41} (r_{12} + r_{23} + r_{34} + r_{41})^5}, \tag{7.26}$$

which agrees with the 0-derivative contribution in the massless case as given in Eq. (3.50).

When $m \neq 0$ the spatial dependence of $V_0^{(4)}$ receives contributions from terms proportional to m^5, m^4/z, m^3/z^2, and m/z^4, each multiplying either $K_0(mz)$ or $K_1(mz)$. We note from Eq. (E27) that when $mz \gg 1$, $K_\nu(mz)$ can be approximated by

$$K_\nu(mz) \simeq \sqrt{\frac{\pi}{2mz}} e^{-mz}, \tag{7.27}$$

so that in the asymptotic regime $V_0^{(4)}$ contains terms of the form

$$\sqrt{\frac{\pi m z}{2}} e^{-mz} \left(\frac{m^4}{z}, \frac{m^3}{z^2}, \frac{m^2}{z^3}, \frac{m}{z^4}, \frac{1}{z^5}\right). \tag{7.28}$$

For values of mz where the exponential makes a significant contribution to W one can approximate $\sqrt{\pi m z/2}$ by unity. It follows that for the term proportional to $1/z^5$, which is the origin of the $m = 0$ result in Eq. (7.6), the most important consequence of a nonzero neutrino mass is that

$$\frac{1}{z^5} \to \frac{e^{-mz}}{z^5}, \tag{7.29}$$

as noted in Appendix E. Since $z = (r_{12} + r_{23} + r_{34} + r_{41})$ the effect of the mean value approximation is to replace z by $4R$ so that

$$\frac{1}{r_{12} r_{23} r_{34} r_{41}} \left(\frac{e^{mz}}{z^5}\right) \to \frac{1}{4^5} \frac{1}{R} \left(\frac{e^{-mR}}{R^2}\right)^4 . \tag{7.30}$$

We conclude from Eq. (7.30) that for the contribution to $W^{(4)}$ arising from $1/z^5$, the primary effect of a nonzero neutrino mass is that $1/R^2$ is replaced by $\exp(-mR)/R^2$ as expected.

It is straightforward to show that this result can be generalized to the k-body case. In order G_F^k the contribution which reproduces the $m = 0$ result arises from the terms proportional to E^k, which leads to the function $F_k(z)$ in Eq. (E20). The generalizations of the recurrence relations in Eqs. (E30) and (E32) eventually express $F_k(z)$ in terms of the k-th derivative $K_1^{(k)}(mz)$ of $K_1(mz)$ by using Eq. (E33). These derivatives can be evaluated from Eq. (E35),

$$K_1^{(1)}(z) = -K_0(z) - \frac{1}{z}K_1(z) \tag{7.31}$$

by repeatedly using Eq. (E34). Among the terms that contribute to $K_1^{(k)}$ will be one which arises from successive differentiations of $1/z$, and this produces a term proportional to $(-1)^k k!/z^k$. When all the appropriate factors are included we find

$$\begin{aligned} E^k \to F_k(z) \to (-i)^k F_0^{(k)}(z) &= (-i)^k 2im^{k+1} K_1^{(k)}(mz) \\ &= \frac{(i)^k k!}{z^{k+1}} 2imzK_1(mz) + \cdots \\ &\simeq 2i^{k+1}\frac{k!}{z^{k+1}}\left(\sqrt{\frac{\pi mz}{2}}e^{-mz}\right) + \cdots, \end{aligned} \tag{7.32}$$

where the dots indicate the remaining contributions to $K_1^{(k)}(mz)$. The coefficient of the expression in parentheses is the massless result in Eq. (3.14), while the remaining factor is the leading $m \neq 0$ modification of the massless result in the asymptotic regime. It follows that in the mean value approximation, where $z \simeq kR$, the modification of the massless result can be approximated by the factor

$$\sqrt{\frac{\pi mz}{2}}e^{-mz} \simeq \sqrt{\frac{\pi mkR}{2}}e^{-mkR}. \tag{7.33}$$

When calculating W the contribution from the exponential is dominated by values of the argument near unity, in which case $(\pi mkR/2)^{1/2}$ is also of order unity and can be dropped. We conclude that when $m \neq 0$ the contribution to $U_0^{(k)}$ from the term we are considering in Eq. (4.22) is approximately given by

$$U_0^{(k)} \simeq \frac{2i^k}{\pi R} \left(\frac{G_F a_n e^{-mR}}{2\pi\sqrt{2}R^2} \right)^2 \frac{(k!)^2}{k^{k+2}}. \tag{7.34}$$

The contributions from the other z-dependent terms in Eq. (7.25) can be treated in a similar manner. Since each term contains either $K_0(mz)$ or $K_1(mz)$, both of which are proportional to e^{-mz}, it follows that each contribution to $U_0^{(k)}$ will contain the damping factor $\exp(-mkR)$ which leads to saturation of the neutrino-exchange forces. The contributions to $U_0^{(k)}$ from this factor are of the form

$$\frac{C_k}{R} \left(\frac{G_F a_n e^{-mR}}{R^2} \right)^{k-\alpha-\beta} \left(\frac{G_F a_n m e^{-mR}}{R} \right)^{\alpha} \left(G_F a_n m^2 e^{-mR} \right)^{\beta}, \tag{7.35}$$

where α and β are integers, and where C_k is a k-dependent coefficient which can in principle be determined for each such term. In practice this would be not only tedious but also unnecessary, since any combination of such terms leads to roughly the same limit on m. For purposes of deriving the limit on m it is helpful to note that for each k there must be at least one term corresponding to $\alpha = \beta = 0$, since this is the only term which reproduces the known contribution to $U_0^{(k)}$ in the $m = 0$ limit. In the next section we use the preceding results to derive a lower bound on the mass of neutrinos.

VIII. BOUND ON THE NEUTRINO MASS

In this section we derive the actual bound on m which follows from the assumption that the mechanism for resolving the neutrino-exchange energy-density catastrophe is a nonzero value of m. A simple bound on the mass m of any neutrino can be inferred from the observation that if the Compton wavelength of the neutrino were larger than the radius R_{10} of the neutron star, then neutrino-exchange forces would behave as if neutrinos were massless. This leads to the estimate,

$$\frac{\hbar}{mc} \lesssim R_{10} \;\; \Rightarrow \;\; mc^2 \gtrsim \frac{\hbar c}{R_{10}} = 2 \times 10^{-11}\,\mathrm{eV}. \tag{8.1}$$

However, a mass this small would be insufficient to prevent a smaller subvolume of the neutron star from having an unphysically large energy density. Specifically, if the neutrino-exchange energy in a given subvolume were to exceed the known mass (and by extension all

other interaction energies) contained in that subvolume, then the subvolume would become unstable against small perturbations, in analogy to Earnshaw's theorem for electrostatics [56,57]. Following Ref. [56], let $\Phi(\vec{x}_0)$ denote the potential energy of a neutron located in the neutron star at a point $\vec{x}_0$, at which no other sources are present. If the neutron is displaced infinitesimally from $\vec{x}_0$ to $\vec{x}_0 + \delta\vec{x}$, where $\delta\vec{x} = (\delta x_1, \delta x_2, \delta x_3)$, then the change in its potential energy is given by

$$\Delta\Phi \equiv \Phi(\vec{x}_0 + \delta\vec{x}) - \Phi(\vec{x}_0) \simeq \delta\vec{x} \cdot \vec{\nabla}\Phi(\vec{x}_0) + \frac{1}{2}\sum_{i,j=1}^{3} \delta x_i \delta x_j \frac{\partial^2\Phi(\vec{x}_0)}{\partial x_i \partial x_j}. \tag{8.2}$$

For the neutron to be in equilibrium to start with, it must be the case that the net field at $\vec{x}_0$ must vanish, and since this is proportional to $\vec{\nabla}\Phi(\vec{x}_0)$, the term proportional to $\delta\vec{x}$ is zero. By an appropriate choice of coordinate system the remaining term in Eq. (8.2) can be diagonalized so that

$$\Delta\Phi \simeq \frac{1}{2}\left[(\delta x_1)^2 \frac{\partial^2\Phi(\vec{x}_0)}{\partial x_1^2} + (\delta x_2)^2 \frac{\partial^2\Phi(\vec{x}_0)}{\partial x_2^2} + (\delta x_3)^2 \frac{\partial^2\Phi(\vec{x}_0)}{\partial x_3^2}\right]. \tag{8.3}$$

For a displacement $\delta\vec{x} = (\delta l, \delta l, \delta l)$, $\Delta\Phi$ would then be given by

$$\Delta\Phi \simeq \frac{1}{2}(\delta l)^2 \nabla^2\Phi(\vec{x}_0). \tag{8.4}$$

If $\Phi(\vec{x}_0)$ were the electromagnetic potential then $\nabla^2\Phi(\vec{x}_0) = 0$ if there are no sources at $\vec{x}_0$. Since the condition for a stable equilibrium is that $\Delta\Phi > 0$ for any displacement $\delta\vec{x}$, it follows that a collection of charges cannot be in stable equilibrium under purely electrostatic forces, which is Earnshaw's theorem. For many-body neutrino-exchange forces $\nabla^2\Phi(\vec{x}_0)$ is non-zero in general, but it can be either positive or negative depending on the product $(-1)^{N/2}\cos(1/\Gamma_R)$ in Eq. (5.24). Since $\Delta\Phi > 0$ cannot be ensured for an arbitrary distribution of neutrons, it follows from Eq. (8.4) that in general such a distribution is unstable if neutrino-exchange forces are the dominant or exclusive forces present. This is the same situation that obtains for electromagnetism, and hence a similar conclusion follows: In the electrostatic case a stable equilibrium requires the presence of other (non-electrostatic) forces. In the present circumstance neutrino-exchange forces must be similarly balanced by

other forces (e.g., gravitational, strong), but for this to be the case, the magnitude of the neutrino-exchange force must be comparable to that of the other forces. Since the energy density arising from the gravitational or strong forces in any subvolume is smaller than the mass contained in that subvolume, we can then assume that the same must hold true for the contribution from neutrino exchange.

It follows from the preceding discussion that the neutrino-exchange energy $W(r)$ inside a volume of radius $r \leq R_{10}$ must be smaller than the mass $M(r)$ inside that volume. From Eqs. (5.24) and (7.34) we note that for the $\alpha = \beta = 0$ term in Eq. (7.35), the above condition leads to

$$\frac{1}{M(r)}\left[\frac{4}{r}\frac{1}{N(r)}\left(\frac{G_F|a_n|N(r)e^{-mr}}{2\pi\sqrt{2}e^2r^2}\right)^{N(r)}\right] < 1, \tag{8.5}$$

where $\cos(1/\Gamma_R)$ has been approximated by unity. It is convenient to rewrite Eq. (8.5) by introducing the length scale L defined by

$$L \equiv \left(\frac{\sqrt{2}G_F|a_n|\rho}{3e^2}\right)^{-1} = 1.7 \times 10^{-5}\,\text{cm} = \frac{1}{1.1\,\text{eV}}. \tag{8.6}$$

In the approximation of neglecting the binding energy of the neutron star, so that $M(r) \simeq N(r)m_n$, Eq. (8.5) can be rewritten in the form

$$\left(\frac{r}{L}\,e^{-mr}\right)^{N(r)} < \frac{rN^2(r)}{4\ell_n}, \tag{8.7}$$

where $\ell_n \equiv \hbar/m_n c = 2.1 \times 10^{-14}$ cm is the Compton wavelength of the neutron. Taking the logarithm of both sides of Eq. (8.7) then leads to the condition

$$m > \frac{1}{r}\left[\ln\left(\frac{r}{L}\right) - \frac{1}{N(r)}\ln\left(\frac{rN^2(r)}{4\ell_n}\right)\right]. \tag{8.8}$$

Since $N(r) = 4\pi r^3\rho/3$, the right hand side can be expressed directly in terms of $r/L \equiv x$, so that Eq. (8.8) becomes

$$Lm > \frac{1}{x}\left[\ln x - \frac{1}{x^3}\left(2 \times 10^{-23} + 5 \times 10^{-25}\ln x\right)\right]. \tag{8.9}$$

This equation must hold for all values of $r \leq R_{10}$, which is the subvolume radius, and hence for all $x \leq R_{10}/L$. From Eq. (8.7) it follows that when $x = r/L < 1$ the inequality holds

even when $m = 0$, and hence $x < 1$ is uninteresting. For $x > 1$ the coefficient of $1/x^3$ in Eq. (8.9) is small compared to $\ln x$ and hence the inequality becomes

$$Lm \gtrsim \frac{1}{x} \ln x. \tag{8.10}$$

Since this inequality must hold for all x, Lm must exceed the largest value that $(1/x) \ln x$ can assume, which is $1/e$. This gives

$$Lm \gtrsim 1/e, \tag{8.11}$$

and,

$$mc^2 \gtrsim \frac{\sqrt{2} G_F |a_n| \rho}{3e^3} = 0.4\,\mathrm{eV}. \tag{8.12}$$

We note that m is proportional to the product $G_F \rho$ which is the only relevant quantity having the dimensions of mass that can be formed from the available dynamical variables. Since the product $G_F \rho$ also arises in the Mikheyev-Smirnov-Wolfenstein (MSW) mechanism [58,59] for neutrino oscillations in matter, a few comments are in order relating the present work and the MSW effect. The effective energy E_{eff} for a real neutrino of mass m and momentum p propagating through a neutron star is given by [2]

$$E_{eff} \simeq p + m^2/2p + \sqrt{2} G_F \rho, \tag{8.13}$$

and hence real neutrinos can be viewed for some purposes as if they had a small mass $\sqrt{2} G_F \rho$. This heuristic picture helps to explain why the index of refraction for neutrinos differs from unity, in analogy to the index of refraction for light propagating in a medium. However, if we pursue the electromagnetic analogy we note that even in a dielectric medium electrostatic forces arising from the exchange of *virtual* photons still obey Coulomb's law, albeit with an attenuated strength. In the present case, the fact that neutrino exchange would retain its long-range character is significant since this implies that the combinatoric factor entering in Eq. (5.1) would remain as $\binom{N}{k}$. This in turn implies that unless the effective Fermi constant in the medium differed from the vacuum value by a factor $\mathcal{O}(10^{11})$, the neutrino-exchange

energy density problem would still exist. In fact one would not expect G_F to be significantly modified by the presence of a medium because there is no analog for neutrino exchange of a polarization charge in electromagnetism. Stated another way, there is no mechanism for shielding the neutrino-exchange force [17].

Returning to Eq. (8.12) we note that the lower bound applies separately to ν_e, ν_μ, and ν_τ, and is compatible with the upper bounds quoted in Eq. (1.1) for the three neutrino species, as shown in Fig. 6. For ν_e the upper and lower bounds are sufficiently close to suggest that direct evidence for $m_{\nu_e} \neq 0$ could be forthcoming in the foreseeable future. Indeed it may well be the case that the anomalies in the flux of solar neutrinos discussed in Sec. I could be a signal for a non-zero neutrino mass compatible with the bound in Eq. (8.12). In addition, the implication that both ν_μ and ν_τ must also be massive may help to solve the "missing mass" problem discussed earlier.

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APPENDIX A: NOTATION, METRIC CONVENTIONS, AND STANDARD MODEL COUPLINGS

In this Appendix we summarize our metric conventions and those for the Dirac equation. We have employed the Pauli metric conventions of Akhiezer and Berestetskii [34], deWit and Smith [60], Lurié [61], and Sakurai [62]. Reference [60] contains an excellent summary of these conventions, along with tables relating the Pauli metric conventions to those of Bjorken and Drell [63] who use real 4-vector notation. In the Pauli conventions, the Dirac equation in configuration space for a particle of mass m is given by ($c = \hbar = 1$),

$$(\gamma \cdot \partial + m)\psi(x) = 0, \tag{A1}$$

where the Dirac matrices $\gamma_\mu = \gamma_\mu^\dagger$ satisfy

$$\begin{aligned} &\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \qquad\qquad \{\gamma_5, \gamma_\mu\} = 0, \\ &\gamma_5 = \gamma_5^\dagger = \gamma_1\gamma_2\gamma_3\gamma_4 = (1/4!)\epsilon_{\mu\nu\lambda\rho}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho. \end{aligned} \tag{A2}$$

Here $\epsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric permutation symbol, and the dagger denotes the Hermitian adjoint. In discussing the many-body contributions from systems containing an odd number of particles, the following trace identities involving $\epsilon_{\mu\nu\lambda\rho}$ are useful (tr denotes the trace over Dirac indices):

$$\begin{aligned} \mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\gamma_5) &= 4\epsilon_{\mu\nu\lambda\rho}, \\ \mathrm{tr}(\gamma_\rho\gamma_\sigma\gamma_\tau\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_5) &= 4(\delta_{\rho\sigma}\epsilon_{\tau\mu\nu\lambda} - \delta_{\rho\tau}\epsilon_{\sigma\mu\nu\lambda} + \delta_{\sigma\tau}\epsilon_{\rho\mu\nu\lambda} \\ &\qquad + \delta_{\mu\nu}\epsilon_{\rho\sigma\tau\lambda} - \delta_{\mu\lambda}\epsilon_{\rho\sigma\tau\nu} + \delta_{\nu\lambda}\epsilon_{\rho\sigma\tau\mu}), \end{aligned} \tag{A3}$$

where $\rho, \sigma, \tau, \mu, \nu, \lambda = 1, 2, 3, 4$. Other useful trace identities are given in Ref. [60].

For purposes of deriving the Schwinger formula [18,28,29] for the weak energy W in Appendix B, the effective low-energy Lagrangian describing the coupling of neutrinos to quarks and leptons is required. Using Ref. [6] the neutrino-quark interaction is given by

$$\mathcal{L}_I^{\nu q} = \frac{G_F}{\sqrt{2}}\ell_\mu(x)\sum_j\left[\epsilon_L(j)i\bar{q}_j(x)\gamma_\mu(1+\gamma_5)q_j(x) + \epsilon_R(j)i\bar{q}_j(x)\gamma_\mu(1-\gamma_5)q_j(x)\right], \tag{A4}$$

$$\ell_\mu(x) = i\bar{\psi}(x)\gamma_\mu(1+\gamma_5)\psi(x), \tag{A5}$$

where $\psi(x)$ and $q_j(x)$ are the field operators for the neutrino and for quark species j respectively. In the absence of radiative corrections the parameters $\epsilon_L(j)$ and $\epsilon_R(j)$ are given in terms of the weak mixing angle θ_W by [6]

$$\begin{aligned} \epsilon_L(u) &= \frac{1}{2} - \frac{2}{3}\sin^2\theta_W \qquad & \epsilon_R(u) &= -\frac{2}{3}\sin^2\theta_W, \\ \epsilon_L(d) &= -\frac{1}{2} + \frac{1}{3}\sin^2\theta_W \qquad & \epsilon_R(d) &= \frac{1}{3}\sin^2\theta_W. \end{aligned} \tag{A6}$$

The neutrino-electron coupling can be similarly expressed as

$$\mathcal{L}_I^{\nu e}(x) = \frac{G_F}{\sqrt{2}}\ell_\mu(x)\left[i\bar{e}(x)\gamma_\mu\left(g_V^{\nu e} + g_A^{\nu e}\gamma_5\right)e(x)\right], \tag{A7}$$

where $e(x)$ is the electron field operator, and the constants $g_V^{\nu e}$ and $g_A^{\nu e}$ are given by

$$g_V^{\nu e} = \frac{1}{2} + 2\sin^2\theta_W \qquad\qquad g_A^{\nu e} = -\frac{1}{2}. \tag{A8}$$

Note that in (A8) the charged-current contribution has been included along with that from the neutral current [64].

For present purposes, we are interested in the coupling of neutrinos to a static source of unpolarized neutrons, protons, or electrons. Since the only relevant contributions in this circumstance are proportional to $\bar{q}_j\gamma_4 q_j$ and $\bar{e}\gamma_4 e$, the effective vector charges a_i describing the couplings to neutrons, protons, and electrons are

$$a_n = 2[\epsilon_L(d) + \epsilon_R(d)] + [\epsilon_L(u) + \epsilon_R(u)] = -\frac{1}{2}, \tag{A9a}$$

$$a_p = 2[\epsilon_L(u) + \epsilon_R(u)] + [\epsilon_L(d) + \epsilon_R(d)] = \frac{1}{2} - 2\sin^2\theta_W = 0.036, \tag{A9b}$$

$$a_e = g_V^{\nu e} = \frac{1}{2} + 2\sin^2\theta_W = 0.964 \tag{A9c}$$

From Table 26.2 of Ref. [6] we note that the agreement between theory and experiment for the Standard Model couplings in Eqs. (A7)–(A9) is typically at the level of a few percent. It follows from the preceding discussion that the effective low-energy neutrino-neutron coupling can be written in the form given in Eq. (B14) below.

APPENDIX B: THE SCHWINGER FORMULA FOR W

As discussed in Sec. II, our derivation of the k-body ($k = 1, \ldots, N$) neutrino-exchange potential utilizes the formalism developed by Hartle for the 4-body case [18,29], which follows in turn from the Schwinger formula [28], Eq. (B34) below. To help clarify the assumptions, notation, and metric conventions that underlie our results, we present here a derivation of the Schwinger formula due to Hartle [29]. Other useful results can be found in Ref. [65] which deals with the related question of effective Lagrangians in quantum electrodynamics.

We are interested in computing the weak-interaction energy W of a collection of N particles (e.g. a nucleus or a neutron star) due to neutrino exchange. Following the discussion in the Introduction, W can be viewed as the weak-energy analog of the static Coulomb energy W_C of a collection of electric charges, which for a nucleus can be approximated by the 2-body contribution in Eq. (2.6). As we have noted previously, one of the novel features of neutrino-exchange is that W is dominated by many-body contributions. We have shown in Sec. V that if W is expressed in the form

$$W = \sum_{k=2}^{N} W^{(k)}, \tag{B1}$$

where $W^{(k)}$ is the k-body contribution, then the dominant contributions to W arise from terms with $k \simeq N$.

Consider for the sake of concreteness an idealized non-rotating spherical neutron star containing N neutrons. To derive the Schwinger formula for W, the weak interaction energy can be viewed (to lowest order in the Fermi constant G_F) as the energy difference between a neutrino propagating in the "vacuum" $|\hat{0}\rangle$ inside the neutron star, and one propagating in the usual matter-free vacuum $|0\rangle$. Thus,

$$W = \langle\hat{0}|H|\hat{0}\rangle - \langle 0|H_0|0\rangle \equiv \mathcal{E} - \mathcal{E}_0, \tag{B2}$$

where H_0 is the free Hamiltonian for the propagating neutrino, and H includes the interactions with the neutrons. If $\psi(x)$ denotes the field operator for the interacting neutrino then

$$H = \int d^3x \mathcal{H}(x) = \int d^3x \ \pi(x)\dot{\psi}(x) = i \int d^3x \ \psi^\dagger(x)\partial_t\psi(x), \tag{B3}$$

where $\partial_t \equiv \partial/\partial t$. From Eqs. (B2) and (B3), we then have ($\bar{\psi} = \psi^\dagger\gamma_4$)

$$\mathcal{E} = i \int d^3x \ \langle\hat{0}|\bar{\psi}(x)\gamma_4\partial_t\psi(x)|\hat{0}\rangle, \tag{B4}$$

and an analogous formula for $\mathcal{E}_0$. We can express $\mathcal{E}$ in terms of the neutrino propagator by writing Eq. (B4) in the form

$$\begin{aligned} \mathcal{E} &= i \int d^3x \left\{\partial_t\langle\hat{0}|\bar{\psi}(x')\gamma_4\psi(x)|\hat{0}\rangle\right\}_{x'\to x} \\ &= i \int d^3x \left\{(\gamma_4)_{\alpha\beta}\partial_t\langle\hat{0}|\bar{\psi}_\alpha(x')\psi_\beta(x)|\hat{0}\rangle\right\}_{x'\to x}, \end{aligned} \tag{B5}$$

where α and β are spinor indices and ∂_t acts only on unprimed variables. We can assume without loss of generality that $t' > t$ in which case,

$$\begin{aligned} \bar{\psi}_\alpha(x')\psi_\beta(x) &= \theta(t'-t)\bar{\psi}_\alpha(x')\psi_\beta(x) - \theta(t-t')\psi_\beta(x)\bar{\psi}_\alpha(x') \\ &\equiv \mathrm{T}[\bar{\psi}_\alpha(x')\psi_\beta(x)] = -\mathrm{T}[\psi_\beta(x)\bar{\psi}_\alpha(x')]. \end{aligned} \tag{B6}$$

If we define the neutrino propagator $S_F(x,x')$ by

$$[S_F(x,x')]_{\beta\alpha} = \langle\hat{0}|\mathrm{T}[\psi_\beta(x)\bar{\psi}_\alpha(x')]|\hat{0}\rangle, \tag{B7}$$

then $\mathcal{E}$ can be cast in the form

$$\mathcal{E} = -i \int d^3x \ \{\partial_t \mathrm{tr}[\gamma_4 S_F(x,x')]\}_{x'\to x} \tag{B8}$$

where tr denotes the trace over Dirac indices. Following Schwinger we introduce the Fourier transform $S_F(\vec{x},\vec{x}',E)$ defined by

$$S_F(x,x') = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iE(t-t')} S_F(\vec{x},\vec{x}',E). \tag{B9}$$

Combining Eqs. (B8) and (B9) we have

$$\begin{aligned} \mathcal{E} &= -i \int d^3x \left\{\partial_t \mathrm{tr}\left[\gamma_4 \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iE(t-t')} S_F(\vec{x},\vec{x}',E)\right]\right\}_{x'\to x} \\ &= -\frac{1}{2\pi} \int d^3x \left\{\mathrm{tr}\left[\int_{-\infty}^{\infty} dE \ E\gamma_4 S_F(\vec{x},\vec{x}',E)\right]\right\}_{\vec{x}'\to\vec{x}}, \end{aligned} \tag{B10}$$

where the limit $t' \to t$ has been taken following the action of ∂_t. If the order of integration with respect to $\vec{x}$ and E is interchanged then

$$\begin{aligned}
\mathcal{E} &= \frac{-1}{2\pi}\int_{-\infty}^{\infty} dE\ E \int d^3x\ \left\{\mathrm{tr}[\gamma_4 S_F(\vec{x},\vec{x}',E)]\right\}_{\vec{x}'\to\vec{x}} \\
&= \frac{-1}{2\pi}\int_{-\infty}^{\infty} dE\ E\left\{\mathrm{Tr}[\gamma_4 S_F(E)]\right\}. \qquad \text{(B11)}
\end{aligned}$$

In the last step of Eq. (B11) we have introduced the operator $S_F(E)$ whose matrix elements give the c-number function $S_F(\vec{x},\vec{x}',E)$:

$$\langle\vec{x}|S_F(E)|\vec{x}'\rangle = S_F(\vec{x},\vec{x}',E). \qquad \text{(B12)}$$

The limit $\vec{x}' \to \vec{x}$ followed by $\int d^3x$ can then be viewed as a formal trace in configuration space, so that in the notation of Eq. (B11) we can write symbolically,

$$\mathrm{Tr} \equiv \mathrm{tr} \times \int d^3x \qquad \text{(B13)}$$

The expression for $\mathcal{E}$ in Eq. (B11) can be recast into a more useful form by expressing the interacting neutrino propagator $S_F(E)$ in terms of the free propagator $S_F^{(0)}(E)$. Following the discussion in Appendix A, we assume that the low-energy coupling of the neutrons and neutrinos can be expressed in terms of the effective Lagrangian density

$$\mathcal{L}_I(x) = \frac{G_F}{\sqrt{2}} a_n N_\mu(x)\ell_\mu(x). \qquad \text{(B14)}$$

Here $N_\mu(x)$ is the neutron current, $\ell_\mu(x)$ is the neutrino current,

$$\ell_\mu(x) = i\bar{\psi}(x)\gamma_\mu(1+\gamma_5)\psi(x), \qquad \text{(B15)}$$

and $a_n = -1/2$. From Eqs. (B14) and (A1) the complete Lagrangian density for neutrinos is

$$\begin{aligned}
\mathcal{L}(x) &= \mathcal{L}_0(x) + \mathcal{L}_I(x) \\
&= -\bar{\psi}(x)[\gamma\cdot\partial + m]\psi(x) + \frac{G_F}{\sqrt{2}} a_n N_\mu(x)\ell_\mu(x), \qquad \text{(B16)}
\end{aligned}$$

where m is the neutrino mass, which we will assume to be zero at this stage. The equation of motion for $\psi(x)$ can be obtained from the Euler-Lagrange equation,

$$\frac{\partial}{\partial x_\lambda}\left[\frac{\partial \mathcal{L}}{\partial(\partial\bar{\psi}/\partial x_\lambda)}\right] - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0, \tag{B17}$$

and is given by

$$\left[\gamma \cdot \partial - \frac{iG_F a_n}{\sqrt{2}}\gamma \cdot N(1+\gamma_5)\right]\psi(x) = 0. \tag{B18}$$

It follows from Eq. (B17) that $S_F(\vec{x}, \vec{x}', E)$ is a solution of the equation

$$\left[\gamma \cdot \eta - \gamma \cdot \tilde{N}\right] S_F(\vec{x}, \vec{x}', E) = -i\delta^3(\vec{x} - \vec{x}'), \tag{B19}$$

where

$$\tilde{N}_\mu \equiv \frac{iG_F a_n}{\sqrt{2}} N_\mu(1+\gamma_5), \tag{B20}$$

$$\gamma \cdot \eta \equiv \vec{\gamma} \cdot \vec{\partial} - \gamma_4 E, \tag{B21}$$

and $\vec{\partial} \equiv \partial/\partial\vec{x}$. If the state vectors are normalized such that

$$\langle \vec{x}|\vec{x}'\rangle = \delta^3(\vec{x} - \vec{x}'), \tag{B22}$$

then Eqs. (B18) and (B12) lead to

$$[\gamma \cdot \eta - \gamma \cdot \tilde{N}]S_F(E) = -iI, \tag{B23}$$

where I is the identity operator. Multiplying both sides of Eq. (B23) by γ_4 gives

$$[E - H]S_F(E) = i\gamma_4, \tag{B24}$$

where

$$H = -i\vec{\alpha} \cdot \vec{\partial} - \gamma_4\gamma \cdot \tilde{N} = H_0 - \gamma_4\gamma \cdot \tilde{N}, \tag{B25}$$

using $\vec{\alpha} = i\gamma_4\vec{\gamma}$. It follows that

$$S_F(E) = i[E - H]^{-1}\gamma_4, \tag{B26a}$$

$$S_F^{(0)}(E) = i[E - H_0]^{-1}\gamma_4. \tag{B26b}$$

Combining Eqs. (B11) and (B26a) we can express $\mathcal{E}$ in the form

$$\mathcal{E} = \frac{-i}{2\pi}\int_{-\infty}^{\infty} dE\ E\left\{\mathrm{Tr}\left(\frac{1}{E-H}\right)\right\}, \tag{B27}$$

and carrying out a partial integration allows Eq. (B27) to be written as

$$\mathcal{E} = \frac{-i}{2\pi}\mathrm{Tr}\left\{E\ln(E-H)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dE\ln(E-H)\right\}. \tag{B28}$$

Since $\mathcal{E}_0$ can be obtained from Eq. (B28) by substituting H_0 for H, we find for W

$$\begin{aligned} W &= \mathcal{E} - \mathcal{E}_0 \\ &= \frac{-i}{2\pi}\mathrm{Tr}\left\{E\ln\left(\frac{E-H}{E-H_0}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dE\ln\left(\frac{E-H}{E-H_0}\right)\right\} \quad . \\ &= \frac{i}{2\pi}\mathrm{Tr}\left\{\int_{-\infty}^{\infty} dE\ln\left(\frac{E-H}{E-H_0}\right)\right\} \quad . \end{aligned} \tag{B29}$$

Using Eqs. (B24) and (B26b) the remaining term in Eq. (B29) can be written as

$$\frac{E-H}{E-H_0} = 1 + \frac{\gamma_4\gamma\cdot\ \tilde{N}}{E-H_0} = 1 - i\gamma_4\gamma\cdot\ \tilde{N}\ S_F^{(0)}(E)\gamma_4. \tag{B30}$$

Combining Eqs. (B29) and (B30) gives

$$W = \frac{i}{2\pi}\mathrm{Tr}\left\{\int_{-\infty}^{\infty} dE\ \ln[1 - i\gamma_4\gamma\cdot\ \tilde{N}\ S_F^{(0)}(E)\gamma_4]\right\}. \tag{B31}$$

Since the integrand represents the infinite series

$$\ln(1-\Delta) = -\sum_{k=1}^{\infty}\frac{\Delta^k}{k}, \tag{B32}$$

it follows that each term in the series will be of the form

$$\mathrm{tr}[\gamma_4\gamma\cdot\tilde{N}(-i)S_F^{(0)}(E)\cdots\gamma_4] = \mathrm{tr}[\gamma\cdot\tilde{N}(-i)S_F^{(0)}(E)\cdots], \tag{B33}$$

using the cyclic property of the trace. The Dirac matrices γ_4 can thus be dropped from Eq. (B31) which then leads via Eq. (B19) to the Schwinger formula for W [18,28]:

$$W = \frac{i}{2\pi}\mathrm{Tr}\left\{\int_{-\infty}^{\infty} dE\ \ln[1 + \frac{G_F a_n}{\sqrt{2}}N_\mu\gamma_\mu(1+\gamma_5)S_F^{(0)}(E)]\right\}. \tag{B34}$$

Following Ref. [29], we show in Sec. III that the Schwinger formula leads directly to a finite expression for the 2-body potential $V^{(2)}(r_{12})$, without having to resort to the regularization methods employed by either FS [15] or HS [17]. Evidently the Schwinger formula must also incorporate a regularization procedure (since the full 2-body amplitude for neutrino-exchange is divergent), but this regularization is built in at the outset when W is expressed as the difference $(\mathcal{E} - \mathcal{E}_0)$ in Eq. (B2).

As noted at the beginning of this Appendix, W is the analog of the Coulomb energy W_C in Eq. (2.6) in the sense of incorporating both the integration over the charge distribution and the combinatorics associated with these charges. For example, when Eq. (B34) is expanded to $\mathcal{O}(G_F^4)$ as in Eq. (3.36), the integrations over all space are explicitly indicated, and the combinatoric factors arise from counting the number of ways that N particles can be assigned to the coordinates $x_1, \ldots, x_4$ in $T_{\mu\nu\lambda\sigma}(x_1, x_2, x_3, x_4)$. In practice we will not calculate W directly from Eq. (B34), but rather use the Schwinger formula as a generating functional to obtain the k-body potentials $V^{(k)}(r_{12}, \ldots, r_{k1})$, as in Eqs. (3.16), (3.50), (3.53), and (3.55). These potentials will then be integrated over the spherical volume in Section IV to produce the $U^{(k)}$, by adapting some formalism from geometric probability. The final expression for U is then obtained in Sec. V by supplying the combinatoric factor $\binom{N}{k}$, which allows us to write

$$W = \sum_{\substack{k=2 \\ even}}^{N} W^{(k)} = \sum_{\substack{k=2 \\ even}}^{N} U^{(k)} \binom{N}{k}. \tag{B35}$$

In order to apply the Schwinger formula in Eq. (B34) it is necessary to exhibit the explicit functional form of the neutrino propagator $S_F^{(0)}(E)$. From Eq. (B7) we see that

$$S_F^{(0)}(x, x') = \langle 0 | \mathrm{T}[\psi(x)\bar{\psi}(x')] | 0 \rangle, \tag{B36}$$

where our conventions for $S_F^{(0)}$ (including various factors of i) follow those of Lurié [61]. Since $S_F^{(0)}(x, x')$ is translationally invariant, we can set $x' = 0$ without loss of generality in which case [61]

$$S_F^{(0)}(x, 0) = S_F^{(0)}(x) = -(\gamma \cdot \partial - m)\Delta_F^{(0)}(x) \xrightarrow{m=0} -\gamma \cdot \partial \Delta_F^{(0)}(x). \tag{B37}$$

Here $\Delta_F^{(0)}(x)$ is the free propagator for a (massless) scalar field, which is given in configuration space by

$$\Delta_F(x) = \frac{1}{4\pi^2(x^2 + i\epsilon)}. \tag{B38}$$

Inverting Eq. (B9) we then have

$$\begin{aligned} S_F^{(0)}(\vec{x}, E) &= \int_{-\infty}^{\infty} dt\ e^{iEt} S_F^{(0)}(x) \\ &= [\vec{\gamma} \cdot \vec{\partial} - \gamma_4 E] \int_{-\infty}^{\infty} dt\ e^{iEt} \frac{(-1)}{4\pi^2(r^2 - t^2 + i\epsilon)}. \end{aligned} \tag{B39}$$

The integral in Eq. (B39) can be evaluated using contour integration, by closing the contour in the upper half-plane for $E > 0$, and in the lower half-plane for $E < 0$. The result is

$$S_F^{(0)}(\vec{x}, E) = \gamma \cdot \eta \left[\frac{i}{4\pi} \frac{e^{i|E|(|\vec{x}| + i\epsilon)}}{|\vec{x}| + i\epsilon} \right], \tag{B40}$$

where

$$\gamma \cdot \eta = \vec{\gamma} \cdot \vec{\partial} - \gamma_4 E. \tag{B41}$$

To describe the propagation of a neutrino from $\vec{r}_j$ to $\vec{r}_i$ where $\vec{r}_i$ and $\vec{r}_j$ are the coordinates of two neutrons, let $\vec{x} \to \vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, with $r_{ij} = |\vec{r}_{ij}|$. Thus the explicit expression for $S_F^{(0)}(\vec{r}_{ij}, E)$ to be used in evaluating the many-body contributions is

$$\begin{aligned} S_F^{(0)}(\vec{r}_{ij}, E) &= [\vec{\gamma} \cdot \vec{\partial}_{ij} - \gamma_4 E] \left[\frac{i}{4\pi} \frac{e^{i|E|(r_{ij} + i\epsilon)}}{r_{ij} + i\epsilon} \right] \\ &\equiv \gamma \cdot \eta(ij) \Delta_F(\vec{r}_{ij}, E), \end{aligned} \tag{B42}$$

where $\vec{\partial}_{ij} \equiv \partial/\partial \vec{r}_{ij}$. In the Tr notation of Eqs. (B13) and (B34), the propagator $S_F^{(0)}(\vec{r}_{ij}, E)$ is to be thought of as the ij matrix element of the operator $S_F^{(0)}(E)$ in Eq. (B34). We note that when $\ln[1 + \cdots]$ in Eq. (B34) is expanded using Eq. (B32), the integrand in Eq. (B34) will contain a polynomial in E in which odd powers of E can be dropped due to the symmetric integration limits.

APPENDIX C: PROBABILITY DISTRIBUTIONS FOR POINTS IN A SPHERICAL VOLUME

As noted in Sec. II, the average value $\langle g \rangle$ of a function $g(r)$ taken over a 3-dimensional spherical volume of radius R is

$$\langle g \rangle = \int_0^{2R} dr\, \mathcal{P}_3(r) g(r), \tag{C1}$$

where $r = r_{12} = |\vec{r}_1 - \vec{r}_2|$ is the separation of two points, and $\mathcal{P}_3(r)$ is the probability distribution. The functional form of $\mathcal{P}_n(r)$ for an n-dimensional ball has been discussed by a number of authors whose work is summarized in Refs. [66] and [67]. It is convenient to introduce the scaled dimensionless variable $s = r/2R$ which satisfies $1 \geq s \geq 0$, and to re-express $\mathcal{P}_3(r)$, $g(r)$, and $\langle g \rangle$ in terms of $\mathcal{P}_3(s)$ and $g(s)$ so that

$$\langle g \rangle = \int_0^{1} ds\, \mathcal{P}_3(s) g(s). \tag{C2}$$

$\mathcal{P}_n(s)$, which denotes the probability that the scaled distance between two points in an n-dimensional ball will be in the interval $(s,\, s + ds)$ is then given by [66,67]

$$\mathcal{P}_n(s) = 2^n n s^{n-1} I_{1-s^2}[(n+1)/2, 1/2]. \tag{C3}$$

$I_x(p, q)$ is the incomplete beta function defined by

$$I_x(p,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^x dt\, t^{p-1}(1-t)^{q-1}. \tag{C4}$$

The results for $n = 1, 2, 3$ are of particular interest in physics, and the corresponding probability distributions are given by [66,67]

$$\mathcal{P}_1(s) = 2(1-s), \tag{C5}$$

$$\mathcal{P}_2(s) = \frac{16}{\pi} s \left[\cos^{-1} s - s(1-s^2)^{1/2}\right], \tag{C6}$$

$$\mathcal{P}_3(s) = 12 s^2 (1-s)^2 (2+s). \tag{C7}$$

The expression for $\mathcal{P}_3(s)$ in Eq. (C7) has been obtained independently by Overhauser [68].

The following properties of $\mathcal{P}_n(s)$ will be useful in the ensuing discussion:

$$\int_0^1 ds\, \mathcal{P}_n(s) = 1, \tag{C8}$$

$$\mathcal{P}_n(1) = 0,\ n \geq 1; \qquad \mathcal{P}_n(0) = 0,\ n \geq 2; \tag{C9}$$

$$\mathcal{P}'_n(1) = 0,\ n \geq 2; \qquad \mathcal{P}'_n(0) = 0,\ n \geq 3, \tag{C10}$$

where the primes denote differentiation with respect s. Another useful result is the mean value $\overline{s}(n)$ of the separation of two points in an n-dimensional ball which is given by [66]

$$\overline{s}(n) = \frac{1}{2}\left(\frac{n}{n+1}\right)^2 \frac{\Gamma(n+2)\Gamma(n/2)}{\Gamma(n+3/2)\Gamma((n+1)/2)}. \tag{C11}$$

For present purposes we are interested in $\mathcal{P}_3 \equiv \mathcal{P}(s)$, which is used in Sections II and IV to evaluate the 2-body contributions arising from photon-exchange and neutrino-exchange respectively. As can be seen from Eq. (C7), $\mathcal{P}(1) = \mathcal{P}(0) = 0$, as required by Eq. (C9); Eq. (C10) can be verified by noting that

$$\mathcal{P}'(s) = 12s(1-s)(4-5s-5s^2). \tag{C12}$$

Since both $\mathcal{P}(s)$ and $\mathcal{P}'(s)$ vanish at the endpoints of the physical region ($s = 1$ and $s = 0$), it follows that $\mathcal{P}(s)$ is a steeply falling function of s at the wings of the distribution. The behavior of $\mathcal{P}(s)$ near $s = 0$ and 1 can be verified from Fig. 3 which exhibits $\mathcal{P}(s)$ and $\mathcal{P}'(s)$ in the interval $1 \geq s \geq 0$. Two other quantities of interest are the location s_0 of the maximum of $\mathcal{P}(s)$, and the mean separation $\overline{s}(3)$ of two points in a 3-dimensional sphere. From Eq. (C12) $\mathcal{P}'(s) = 0$ gives local minima for $\mathcal{P}(s)$ (in the physical region $1 \geq s \geq 0$) at $s = 0, 1$ and a local maximum for $\mathcal{P}(s)$ when $(4 - 5s - 5s^2) = 0$. The physical root is then $s_0 = [-1/2 + (21/20)^{1/2}] = 0.5247$ which is close to the middle of the physical region, $s = 0.5$. To evaluate $\overline{s}(3)$ we use Eq. (C11) with $n = 3$ which gives $\overline{s}(3) = 18/35 = 0.5143$. The result $\overline{s}(3) \simeq 0.5$ is understandable given that $\mathcal{P}(s)$ peaks near $s_0 \simeq 0.5$, and the slight difference between $\overline{s}(3)$ and s_0 is a reflection of the fact that $\mathcal{P}(s)$ is not symmetric about $s = 0.5$.

For purposes of averaging the k-body 0-derivative contributions over a spherical volume it is in principle necessary to know the k-body generalization of $\mathcal{P}(s)$ in 3-dimensions, and to be able to integrate this distribution over the coordinates of $k \leq N = \mathcal{O}(10^{57})$ particles. At present the dependence of this function, $\mathcal{P}^{(k)}(s_{ij}) \equiv \mathcal{P}^{(k)}(s_{12}, s_{23}, \ldots, s_{k1})$, on the variables $s_{ij} = |\vec{r}_i - \vec{r}_j|/2R$ is not known. Moreover, even if $\mathcal{P}^{(k)}(s_{ij})$ were known, the task of evaluating the k-body generalization of the integral in Eq. (4.21) would be beyond present computational capabilities. For these reasons we use the preceding discussion of $\mathcal{P}(s)$ to approximate $V^{(k)}(\vec{r}_{12}, \vec{r}_{23}, \ldots, \vec{r}_{k1})$ in Eq. (3.59) by replacing r_{ij} with its mean value,

$$r_{ij} \longrightarrow \langle r_{ij} \rangle = 2R\,\overline{s}(3) \simeq R. \tag{C13}$$

The integrals in Eq. (4.16) can then be carried out as in Eq. (4.17) and give

$$U_0^{(k)} \simeq \frac{4}{\pi} \left(\frac{G_F a_n}{2\pi\sqrt{2}} \right)^k i^k k! \frac{1}{R^k (kR)^{k+1}} \int_0^{2R} dr_{12} \cdots \int_0^{2R} dr_{k1}\, \mathcal{P}^{(k)}(r_{ij}), \tag{C14}$$

where $r_{ij} = 2Rs_{ij}$. Using the normalization condition, Eq. (4.15), we find,

$$U_0^{(k)} \simeq \frac{4}{\pi R} \left(\frac{G_F a_n}{2\pi\sqrt{2}R^2} \right)^k \frac{i^k k!}{k^{k+1}}, \tag{C15}$$

and this is the expression which will be used in Secs. IV and V. The "mean value approximation" in Eq. (C13) can be justified by comparing Eq. (C13) to the actual value of $\langle r_{ij} \rangle$ for the 2-body electromagnetic, 3-body weak, and 4-body weak potentials which have been evaluated directly [35]. We find for these cases, respectively: $\langle r_{ij} \rangle = 0.83R$, $\langle r_{ij} \rangle = 0.48R$, $\langle r_{ij} \rangle = 0.62R$. We note that the mean value approximation in Eq. (C13) overestimates $\langle r_{ij} \rangle$, and hence it *underestimates* $|U_0^{(k)}|$ and ultimately $|W|$ in Eq. (5.37). Moreover, the uncertainties arising from the mean value approximation are no worse than those inherent in estimating R itself [42,69]. The fact that all the values of $\langle r_{ij} \rangle$ are reasonably close to R can be understood intuitively as follows: The constraints in Eqs. (C9) and (C10) serve to suppress $\mathcal{P}(s)$ at the wings of the distribution and, since $\mathcal{P}(s)$ is normalized, the result is a peaking of $\mathcal{P}(s)$ near $s \simeq 0.5$ which corresponds to $r_{ij} \simeq R$.

APPENDIX D: EVALUATION OF $\sum_k W^{(k)}$

From Eq. (5.3) the weak energy W is given by

$$W \simeq W^{(2)} + \sqrt{\frac{2}{\pi}} \frac{2}{R} \sum_{\substack{k=4 \\ even}}^{N} \frac{i^k k!}{k^{3/2}} \left(\frac{G_F a_n}{2\pi\sqrt{2} e R^2} \right)^k \binom{N}{k}. \tag{D1}$$

In Sec. V we approximated $\sum_k$ by replacing $k^{3/2}$ in the denominator of Eq. (D1) by $N^{3/2}$, noting that the sum is dominated by terms with $k \simeq N$. In this Appendix we show how $\sum_k$ can be evaluated with the factor of $k^{3/2}$ included, should a more refined approximation to the sum be desired [70]. Define

$$A_k = i^k k! \binom{N}{k} \qquad\qquad \Gamma_R = \frac{G_F a_n}{2\pi\sqrt{2} e R^2}, \tag{D2}$$

and

$$\sum_k = \sum_k \frac{A_k \Gamma_R^k}{k^{3/2}}. \tag{D3}$$

In Eq. (D3) let $\Gamma_R = e^y$ so that

$$\sum_k = \sum_k \frac{A_k e^{ky}}{k^{3/2}} = \sum_k A_k \frac{1}{\Gamma(3/2)} \int_{-\infty}^{y} dz\, e^{kz} (y-z)^{1/2}. \tag{D4}$$

The last step in Eq. (D4) can be verified by substituting $t = (y-z)$ which allows the integral in Eq. (D4) to be expressed in terms of the gamma function:

$$\int_{-\infty}^{y} dz\, e^{kz} (y-z)^{1/2} = e^{ky} \int_0^{\infty} dt\, e^{-kt} t^{1/2} = \frac{e^{ky}}{k^{3/2}} \Gamma(3/2). \tag{D5}$$

Returning to Eq. (D4) we substitute $z = \ln \omega$ to recast $\sum_k$ into the form

$$\sum_k \frac{A_k e^{ky}}{k^{3/2}} = \frac{1}{\Gamma(3/2)} \sum_k A_k \int_0^{\Gamma_R} d\omega\, \omega^{k-1} (\ln \Gamma_R - \ln \omega)^{1/2}. \tag{D6}$$

Hence,

$$\sum_k \frac{A_k \Gamma_R^k}{k^{3/2}} = \sum_k \frac{A_k e^{ky}}{k^{3/2}} = \frac{1}{\Gamma(3/2)} \int_0^{\Gamma_R} \frac{d\omega}{\omega} (\ln \Gamma_R - \ln \omega)^{1/2} \sum_k A_k \omega^k. \tag{D7}$$

Using Eq. (D2) we see that the expression for $\sum_k$ on the right-hand side of Eq. (D7) now has the same form as the sum previously evaluated in Sec. V starting from Eq. (5.7). Hence by utilizing the result in Eq. (5.20), the sum on the left-hand side of Eq. (D7) can be expressed in terms of a one-dimensional integral, which can be evaluated numerically if necessary.

APPENDIX E: THE SCHWINGER-HARTLE FORMALISM FOR MASSIVE NEUTRINOS

In this Appendix we generalize the Schwinger-Hartle formalism presented in Appendix B to the case of massive neutrinos. Although the Schwinger formula itself retains the same form as in Eq. (B34), the expression for the free neutrino propagator $S_F^{(0)}(E)$ is modified when the neutrino mass m is different from zero. The final expression for the massive propagator $S_{Fm}^{(0)}(E)$ is given in Eq. (E18), and the application of the Schwinger-Hartle formalism to the present problem is discussed in Sec. VII.

To establish our conventions when $m \neq 0$, we begin with the expression for the propagator $\Delta_{Fm}^{(0)}$ of a massive scalar field, which is given in our metric by [71,72]

$$\Delta_{Fm}^{(0)}(x) = -\frac{1}{(2\pi)^3}\int_{C_F} d^3k \int \frac{dk_0}{2\pi i}\,\frac{e^{i(\vec{k}\cdot\vec{x}-k_0x_0)}}{(k_0-\omega_k)(k_0+\omega_k)}. \tag{E1}$$

Here $\omega_k \equiv +(\vec{k}^2+m^2)^{1/2}$ and C_F is the usual Feynman contour in the complex k_0 plane. The integral over k_0 is straightforward and gives

$$\Delta_{Fm}^{(0)}(x) = \frac{1}{2(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\left(\frac{e^{-i\omega_k|x_0|}}{\omega_k}\right). \tag{E2}$$

Since $\Delta_{Fm}^{(0)}(x)$ is an invariant function of $x^2 = \vec{x}^2 - x_0^2$, it will have the same form for spacelike or timelike x^2. In the former case $\Delta_{Fm}^{(0)}(x)$ can be easily evaluated by setting $x_0 = 0$ which gives [72],

$$\Delta_{Fm}^{(0)}(x) = \frac{m}{8i\pi^2 r}\int_{-\infty}^{\infty} dy\, \sinh y e^{imr\sinh y}, \tag{E3}$$

where $r = |\vec{x}|$ and $\sinh y = |\vec{k}|/m$. $\Delta_{Fm}^{(0)}$ can be expressed in terms of the derivative of the Hankel function $H_0^{(2)}$ which has the integral representation [72]

$$H_0^{(2)}(r) = \frac{i}{\pi}\int_{-\infty}^{\infty} dy\, e^{-ir\cosh y}. \tag{E4}$$

It follows from Eq. (E4) that

$$H_0^{(2)}(-ir) = \frac{i}{\pi}\int_{-\infty}^{\infty} dy\, e^{ir\sinh y}, \tag{E5}$$

and hence [72]

$$\Delta_{Fm}^{(0)}(x) = \frac{i}{8\pi r}\frac{d}{dr}H_0^{(2)}(-imr) = \frac{im^2}{8\pi r}\frac{H_1^{(2)}(-imr)}{-imr}. \tag{E6}$$

The expression for $\Delta_{Fm}^{(0)}(x)$ can be written in manifestly covariant form by letting $r \to \sqrt{x^2}$, so that finally [72],

$$\Delta_{Fm}^{(0)}(x) = -\frac{m}{8\pi}\frac{H_1^{(2)}(-im\sqrt{x^2})}{\sqrt{x^2}}. \tag{E7}$$

It is useful to check Eq. (E7) in the two limiting cases of interest to us: a) For $m \to 0$ we use the series expansion [73]

$$H_\nu^{(2)}(z) \simeq \frac{i(\nu-1)!}{\pi}\left(\frac{2}{z}\right)^\nu, \tag{E8}$$

to obtain

$$\Delta_{Fm}^{(0)}(x) \overset{m\to 0}{\longrightarrow} -\frac{m}{8\pi\sqrt{x^2}}\left[\frac{i}{\pi}\left(\frac{2}{-im\sqrt{x^2}}\right)\right] = \frac{1}{4\pi x^2}, \tag{E9}$$

in agreement with Eq. (B38). b) When $|z|$ is large $H_\nu^{(2)}(z)$ can be represented by the asymptotic expansion [74]

$$H_\nu^{(2)}(z) \simeq \left(\frac{2}{\pi z}\right)^{1/2} \exp\left\{-i[z-(\nu+1/2)\pi/2]\right\}. \tag{E10}$$

Hence with $z = -im\sqrt{x^2}$ we have approximately,

$$\Delta_{Fm}^{(0)}(x) \simeq \frac{m^2}{4\sqrt{2}}\left(\frac{1}{\pi m\sqrt{x^2}}\right)^{3/2} e^{-m\sqrt{x^2}}, \tag{E11}$$

which shows the characteristic exponential decrease of $\Delta_{Fm}^{(0)}(x)$. The analogous behavior for neutrinos eventually leads to the "saturation" of the many-body contribution to W, and thus to a resolution of the neutrino-exchange energy-density catastrophe.

The massive fermion propagator in configuration space can be obtained from Eq. (E7) and, as in the massless case, we are interested in calculating

$$S_{Fm}^{(0)}(\vec{x},E) = \int_{-\infty}^{\infty} dt\, e^{iEt} S_{Fm}^{(0)}(x) = (-\gamma\cdot\eta + m)\int_{-\infty}^{\infty} dt\, e^{iEt}\Delta_{Fm}^{(0)}(x), \tag{E12}$$

where $\gamma \cdot \eta = \vec{\gamma} \cdot \vec{\partial} - \gamma_4 E$, and $t = x_0$. We define

$$\Delta_{Fm}^{(0)}(\vec{x}, E) = \int_{-\infty}^{\infty} dt\, e^{iEt} \Delta_{Fm}^{(0)}(x) = \int_{-\infty}^{\infty} dt\, e^{iEt} \left[\left(\frac{im^2}{8\pi} \right) \frac{H_1^{(2)}(-im\sqrt{r^2 - t^2})}{-im\sqrt{r^2 - t^2})} \right]$$

$$= 2 \int_{-\infty}^{\infty} dt\, \cos(Et) \left[\left(\frac{im^2}{8\pi} \right) \frac{H_1^{(2)}(-im\sqrt{r^2 - t^2})}{-im\sqrt{r^2 - t^2})} \right], \tag{E13}$$

where the last step follows by noting that the expression in square brackets is an even function of t. The integral in Eq. (E13) can be evaluated by making use of the relations [72,75]

$$\int_0^{\infty} dt\, H_0^{(2)}(\alpha\sqrt{\beta^2 - t^2}) \cos(\gamma t) = \frac{i \exp(-i\beta\sqrt{\alpha^2 + \gamma^2})}{\sqrt{\alpha^2 + \gamma^2})}, \tag{E14}$$

$$\frac{d}{dz} H_0^{(2)}(z) = -H_1^{(2)}(z). \tag{E15}$$

Differentiating both sides of Eq. (E14) with respect to β, and using Eq. (E15) we find

$$\int_0^{\infty} dt\, \frac{H_1^{(2)}(\alpha\sqrt{\beta^2 - t^2}) \cos(\gamma t)}{\alpha\sqrt{\beta^2 - t^2}} = -\frac{1}{\alpha^2 \beta} \exp(-i\beta\sqrt{\alpha^2 + \gamma^2}). \tag{E16}$$

If we identify $\alpha = -im$, $\beta = -r$, and $\gamma = E$, then

$$\Delta_{Fm}^{(0)}(\vec{x}, E) = \left(\frac{im^2}{8\pi} \right) \left(\frac{-2}{m^2 r} \right) \exp(ir\sqrt{E^2 - m^2}), \tag{E17}$$

and

$$S_{Fm}^{(0)}(\vec{x}, E) = (\vec{\gamma} \cdot \vec{\partial} - \gamma_4 E - m) \frac{i}{4\pi r} \exp(ir\sqrt{E^2 - m^2})$$

$$\equiv (\gamma \cdot \eta - m) \Delta_{Fm}(\vec{r}, E). \tag{E18}$$

In the $m \to 0$ limit the expression in Eq. (E18) reduces to the massless result given in Eq. (B42).

When the Schwinger formula in Eq. (B42) is expanded in a perturbation series, and the Dirac traces are evaluated, integrals of the form

$$\bar{F}_n(z) = \int_{-\infty}^{\infty} dE\, E^n \exp(iz\sqrt{E^2 - m^2}) \tag{E19}$$

arise which are the $m \neq 0$ analogs of $\bar{I}_n(z)$ in Eq. (3.11). Since $\bar{F}_n(z) = 0$ when n is odd we can write

$$\bar{F}_n(z) = \begin{cases} 2\int_0^\infty dE\, E^n \exp(iz\sqrt{E^2 - m^2}) \equiv F_n(z) & \text{even } n \\ 0 & \text{odd } n. \end{cases} \tag{E20}$$

The integrals in Eqs. (E19) and (E20) can be evaluated recursively beginning with $F_0(z)$ which may be cast in the form

$$F_0(z) = 2\int_{im}^\infty dp\, e^{izp} \frac{p}{\sqrt{p^2 + m^2}}, \tag{E21}$$

where $p^2 = E^2 - m^2$. $F_0(z)$ can be expressed in terms of a modified Bessel function by analytically continuing the integral [76]

$$\int_u^\infty dp\, \frac{pe^{-\tau p}}{\sqrt{p^2 - u^2}} = uK_1(\tau u). \tag{E22}$$

Here $K_n(z)$ is the modified Bessel function defined by

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix), \tag{E23}$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind. Substituting $u = im$ and $\tau = iz$ then leads to

$$F_0(z) = 2imK_1(mz). \tag{E24}$$

For small x the leading term in the series expansion of $K_\nu(x)$ is

$$K_\nu(x) \simeq 2^{\nu-1}(\nu - 1)!x^{-\nu}, \tag{E25}$$

and hence for small z

$$F_0(z) \simeq 2im\frac{1}{mz} = \frac{2i}{z}, \tag{E26}$$

in agreement with Eq. (3.12). However, the regime of interest when $m \neq 0$ is when $x = mz$ is large, in which case $K_\nu(x)$ can be approximated by the asymptotic series [74]

$$K_\nu(x) \simeq \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + \frac{(4\nu^2 - 1)}{1!8x} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8x)^2} + \cdots \right]. \tag{E27}$$

Combining Eqs. (E24) and (E27) then leads to

$$F_0(z) \simeq 2im\sqrt{\frac{\pi}{2mz}} e^{-mz} \left[1 + \frac{3}{8mz} + \cdots \right], \tag{E28}$$

which exhibits the characteristic exponentially falling behavior that leads to "saturation" of the neutrino-exchange forces.

To evaluate $F_n(z)$ for $n \neq 0$ we differentiate $F_0(z)$ twice with respect to z:

$$\begin{aligned} \frac{d^2 F_0(z)}{dz^2} \equiv F_0^{(2)} &= i^2 2 \int_0^\infty dE\, (E^2 - m^2) \exp(iz\sqrt{E^2 - m^2}) \\ &= i^2 F_2(z) + m^2 F_0(z), \end{aligned} \tag{E29}$$

and hence

$$F_2(z) = (-i)^2 F_0^{(2)}(z) + m^2 F_0(z). \tag{E30}$$

Proceeding in a similar fashion, we find upon differentiating both sides of Eq. (E29) twice with respect to z,

$$F_4(z) = F_0^{(4)}(z) + 2m^2 F_2(z) - m^4 F_0(z). \tag{E31}$$

Combining Eqs. (E30) and (E31) then leads to

$$F_4(z) = F_0^{(4)}(z) - 2m^2 F_0^{(2)}(z) + m^4 F_0(z). \tag{E32}$$

It follows from Eqs.(E30) and (E32) that $F_n(z)$ can be evaluated recursively for any even n in terms of $F_0(z)$ and its derivatives $F_0^{(2)}(z)$, $F_0^{(4)}(z)$,..., $F_0^{(n)}(z)$.

The expressions for $F_2(z)$, $F_4(z)$,... $F_n(z)$, can be further simplified by explicitly evaluating the various derivatives $F_0^{(n)}(z)$ that arise in the expression for $F_n(z)$. From Eq. (E24),

$$F_0^{(n)}(z) = 2im^{n+1} K_1^{(n)}(mz), \tag{E33}$$

where the superscript (n) denotes the nth derivative on both sides of Eq. (E33). The nth derivative of the modified Bessel function $K_1(x)$ can be expressed in terms of $K_0(x)$ and $K_1(x)$ by utilizing the relations [77]

$$\frac{d}{dz}K_0(z) \equiv K_0^{(1)}(z) = -K_1(z), \tag{E34}$$

$$K_1^{(1)}(z) = -K_0(z) - \frac{1}{z}K_1(z). \tag{E35}$$

Differentiating Eq. (E35) and using Eq. (E34) leads to

$$K_1^{(2)}(z) = \frac{1}{z}K_0(z) + \left(1 + \frac{2!}{z^2}\right) K_1(z). \tag{E36}$$

Proceeding in this way we can express the nth derivative $K_1^{(n)}(z)$ in terms of $K_0(z)$ and $K_1(z)$. For purposes of calculating the 4-body contribution in Sec. VII the explicit expressions for $K_1^{(3)}(z)$ and $K_1^{(4)}(z)$ are needed, and these are given by

$$K_1^{(3)}(z) = -\left(1 + \frac{3}{z^2}\right) K_0(z) - \left(\frac{2}{z} + \frac{3!}{z^3}\right) K_1(z), \tag{E37}$$

$$K_1^{(4)}(z) = \left(\frac{2}{z} + \frac{12}{z^3}\right) K_0(z) + \left(1 + \frac{7}{z^2} + \frac{4!}{z^4}\right) K_1(z). \tag{E38}$$

We note from Eq. (7.24) that when the trace of the 4-body matrix element is calculated the result will depend on the functions $F_0^{(4)}(z)$, $m^2F_0^{(2)}(z)$, and $m^4F_0(z)$ which are given by

$$m^4F_0(z) = 2im^5K_1(mz) \tag{E39a}$$

$$m^2F_0^{(2)}(z) = 2i\left[\frac{m^4}{z}K_0(mz) + \left(m^5 + m^3\frac{2!}{z^2}\right) K_1(mz)\right], \tag{E39b}$$

$$F_0^{(4)}(z) = 2i\left[\left(\frac{2m^4}{z} + \frac{12m^2}{z^3}\right) K_0(mz) + \left(m^5 + \frac{7m^3}{z^2} + m\frac{4!}{z^4}\right) K_1(mz)\right]. \tag{E39c}$$

In the $m = 0$ limit, the only term which survives is the contribution proportional to m/z^4, and since this term is also the source of the bound on m in Eq. (8.12) we examine it in more detail.

As $m \to 0$ we have from Eq. (E25)

$$2im\frac{4!}{z^4}K_1(mz) \simeq 2im\frac{4!}{z^4}\,\frac{1}{mz} = 2i\frac{4!}{z^5}, \tag{E40}$$

in agreement with Eq. (3.14). As before, we are interested in the behavior of this term when $m \neq 0$ and z is large. Using Eq. (E27) we find

$$F_0^{(4)}(z) \simeq 2i\frac{4!}{z^5}e^{-mz}\left[\sqrt{\frac{\pi mz}{2}}\left(1+\frac{3}{8mz}+\cdots\right)\right]. \tag{E41}$$

Since the expression in square brackets is slowly varying (in z) compared to the remaining z-dependent factors, the net effect of having $m \neq 0$ is to replace $1/z^5$ by $\exp(-mz)/z^5$ in $F_0^{(4)}$. This result is the basis for the bound on m derived in Sec. VIII.

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TABLES

TABLE I. Summary of the necessary conditions for the existence of a large many-body effect. For each interaction, "Yes" indicates that the condition is met, and "No" indicates that it is not. Among known interactions only neutrino-exchange meets all three conditions. See text for further details.

Condition	Strong	Electomagnetism	Weak [a]	Gravity	Neutrino Exchange
Long-Range	No	Yes	No	Yes	Yes
Bulk Matter "Charge"	Yes	No	Yes	Yes	Yes
Large Coupling Strength	Yes	Yes	Yes	No	Yes

[a] Z^0-exchange.

TABLE II. Parameters for two typical white dwarfs.

	Sirius B	40 Eri B
M [a]	$1.053(28)M_\odot$	$0.48(2)M_\odot$
R [a]	$0.0074(6)R_\odot$	$0.0124(5)R_\odot$
N_e	6.3×10^{56}	2.9×10^{56}
Λ_R [b]	4.1×10^4	6.6×10^3
$W/M_U c^2$ [c]	$10^{(3\times10^{57}+28-13-88)}$	$10^{(1\times10^{57}+28-13-88)}$

[a] From Ref. [51]

[b] Defined by Eq. (5.54).

[c] M_U is the mass of the universe given by Eq. (5.41).

FIGURES

FIG. 1. The 2-body neutron-neutron potential arising from neutrino-exchange. The solid (dashed) lines denote neutrons (neutrinos).

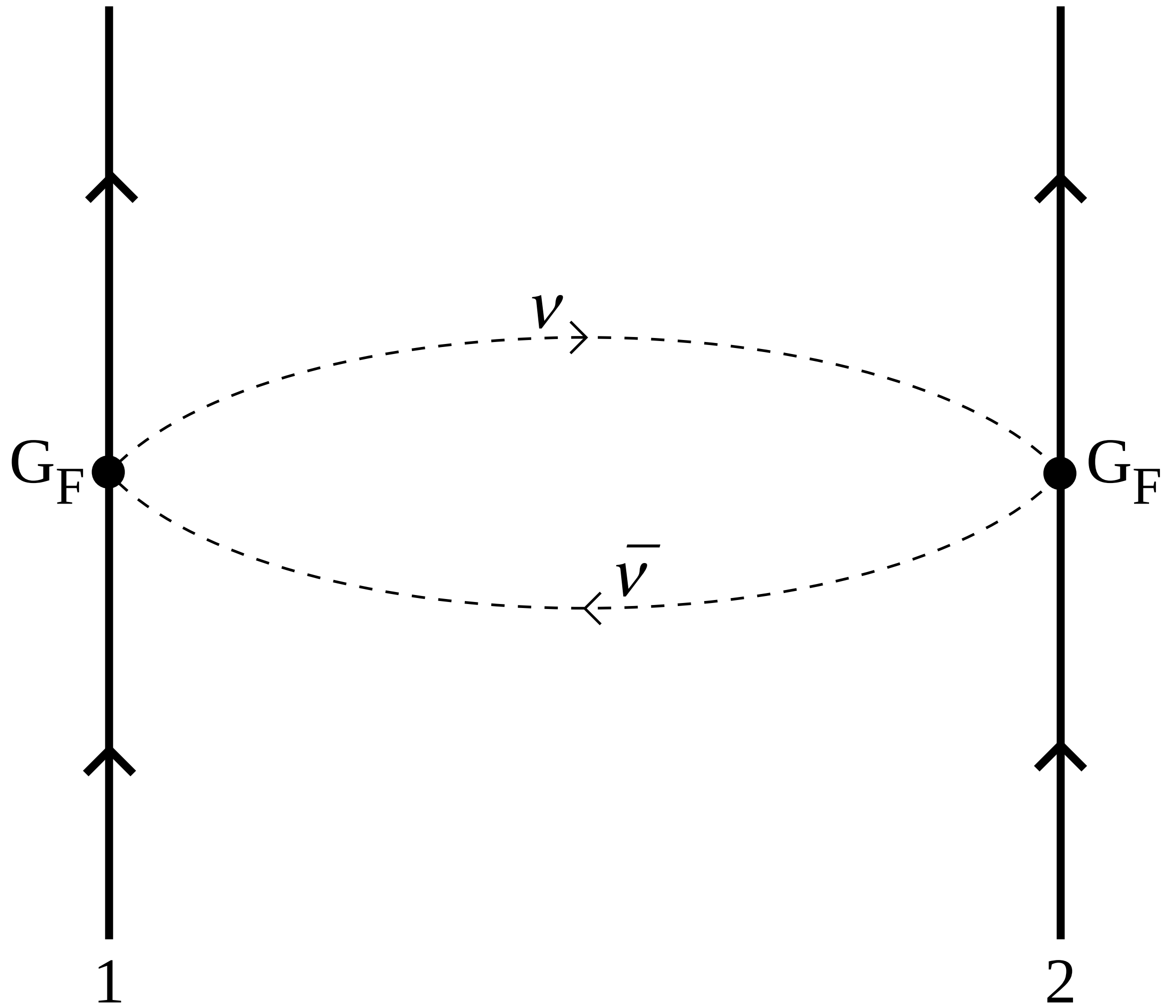
FIG. 2. Contributions to the 4-body potential energy arising from neutrino exchange. As before, solid (dashed) lines denote neutrons (neutrinos). Each of the diagrams (a), (b), and (c) is topologically different from the others, as can be seen by redrawing the graphs as shown. For each of these diagrams there is another that is obtained by reversing the sense of the neutrino loop momentum, as we show explicitly in Fig. 4.

FIG. 3. a) Plot of the function $\mathcal{P}(s) = \mathcal{P}_3(s)$ in Eqs. (2.12) and (C7). b) Plot of $\mathcal{P}'(s)$ in Eq. (C12).

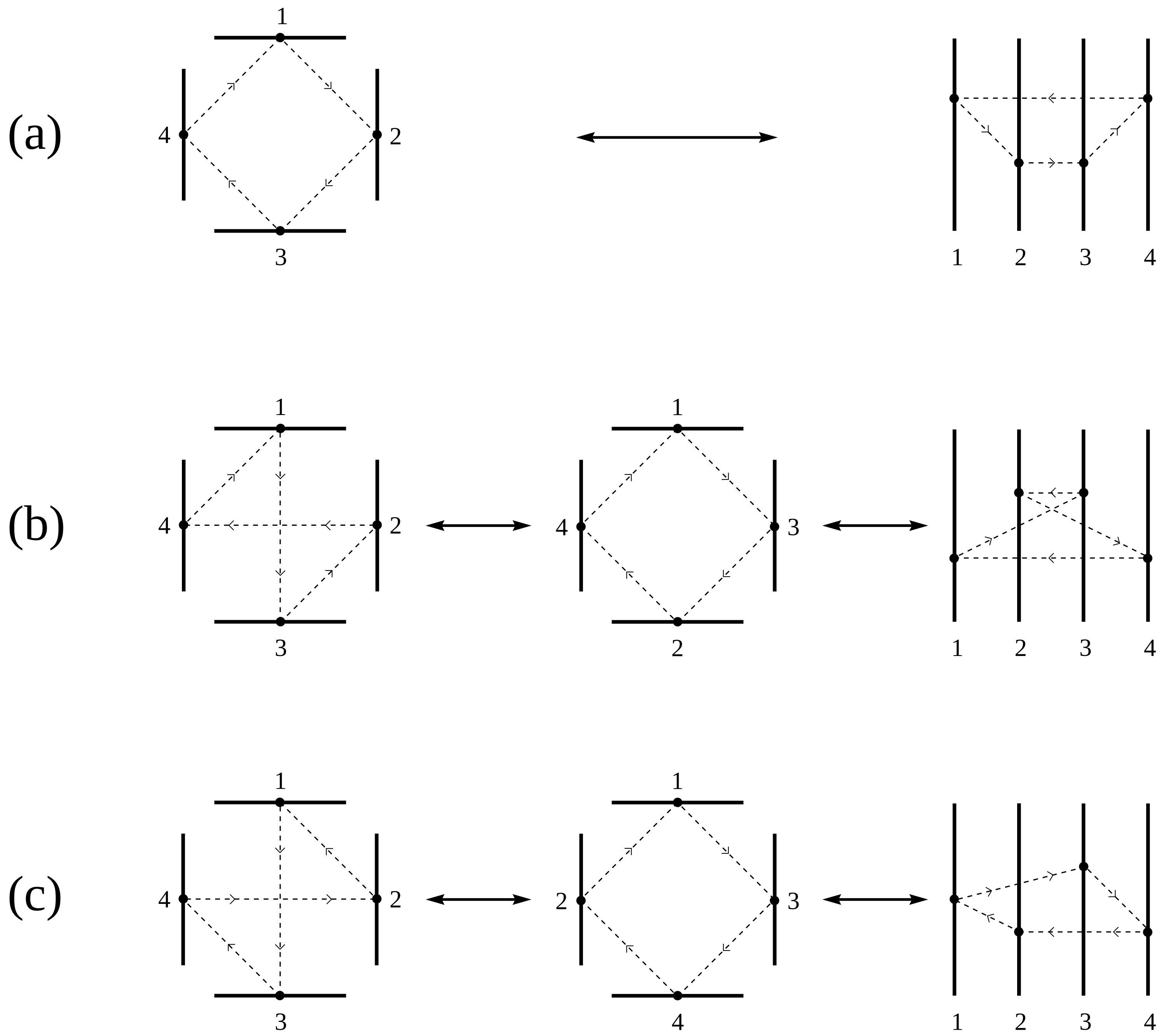
FIG. 4. Combinatorics for the 4-body diagrams. As before, the solid (dashed) lines denote neutrons (neutrinos). For $k = 4$ there are $3! = 6$ topologically distinct diagrams that can be drawn, corresponding to the 6 possible permutations of the integers 1, 2, 3, and 4 arranged at the vertices of the neutrino loop. However, diagrams (a'), (b'), and (c') are obtained from diagrams (a), (b), and (c) respectively by reversing the sense of the neutrino loop momentum. Hence there are $(k-1)!/2$ pairs of diagrams, [(a) + (a')], [(b) + (b')], [(c) + (c')], etc.

FIG. 5. The 3-body contributions arising from neutrino exchange. The solid (dashed) lines represent neutrons (neutrinos). As noted in the text, both diagrams must be included to reproduce Furry's theorem for the vector contribution.

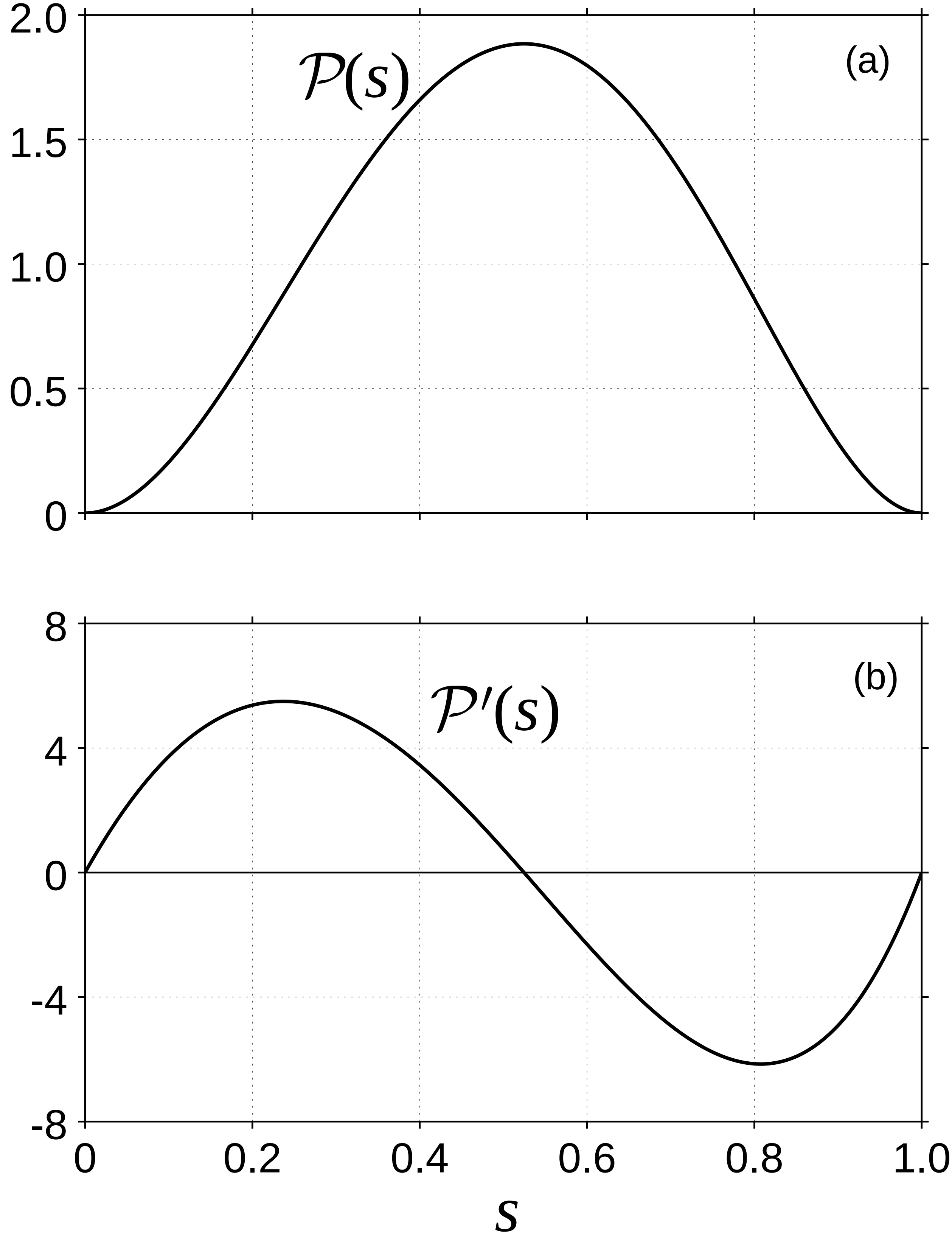
FIG. 6. Constraints on the masses of ν_e, ν_μ, and ν_τ. For each neutrino (or antineutrino) the shaded regions are excluded either by the lower bound in Eq. (8.12) or by the upper bounds in Eq. (1.1).



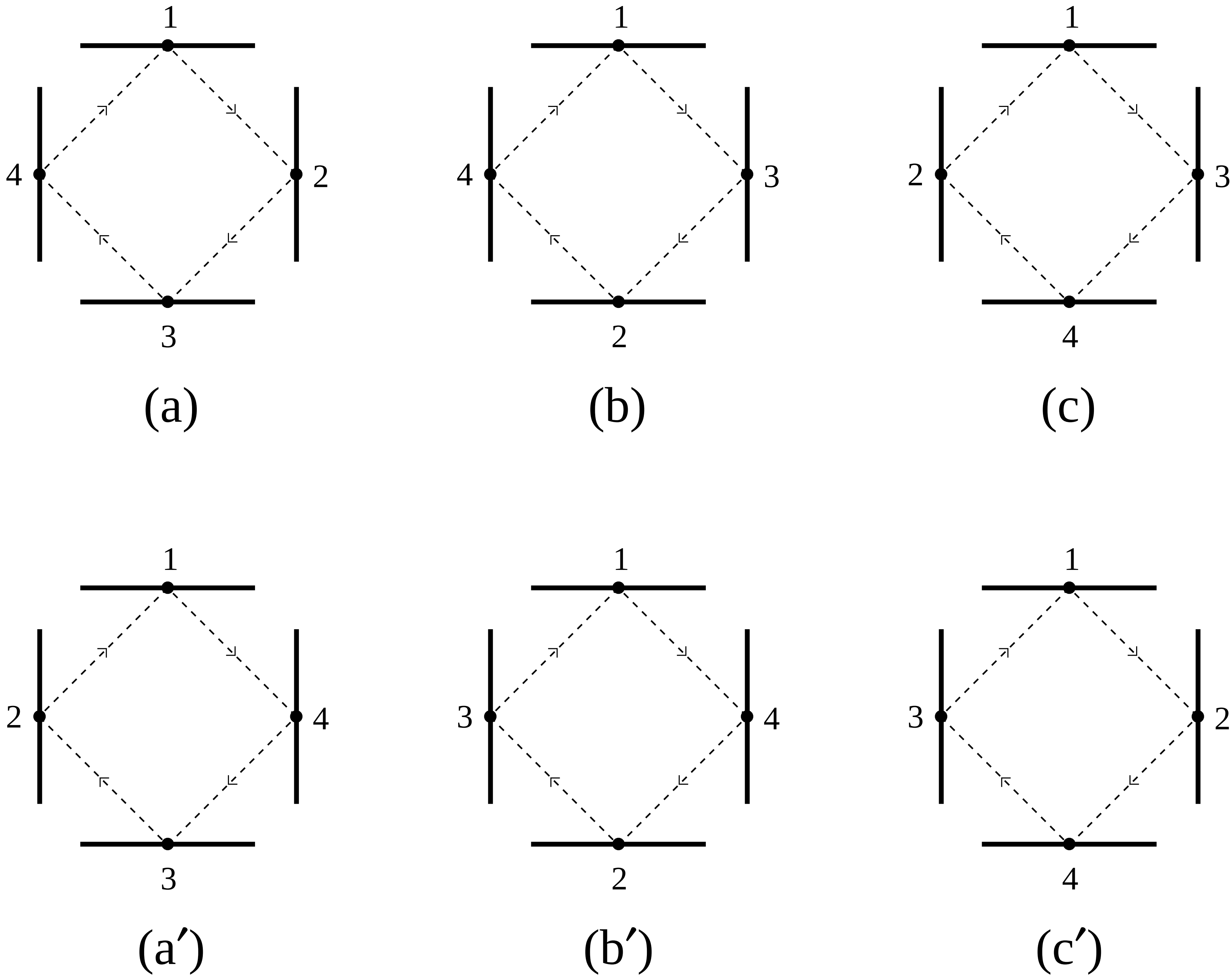
ν
G_F
G_F
$\bar{\nu}$
1
2



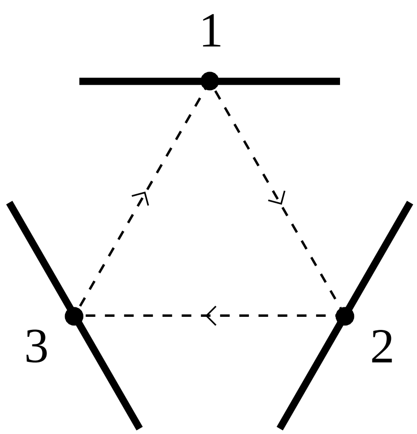
(a)
1
4
2
3
1 2 3 4
(b)
1
4
2
3
1
4
3
2
1 2 3 4
(c)
1
4
2
3
1
2
3
4
1 2 3 4



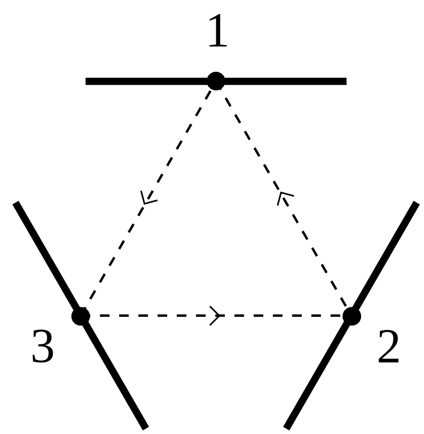
2.0
1.5
1.0
0.5
0
$\mathcal{P}(s)$
(a)
8
4
0
-4
-8
$\mathcal{P}'(s)$
(b)
0
0.2
0.4
0.6
0.8
1.0
s



1
4
2
3
(a)
1
4
3
2
(b)
1
2
3
4
(c)
1
2
4
3
(a′)
1
3
4
2
(b′)
1
3
2
4
(c′)



(a)



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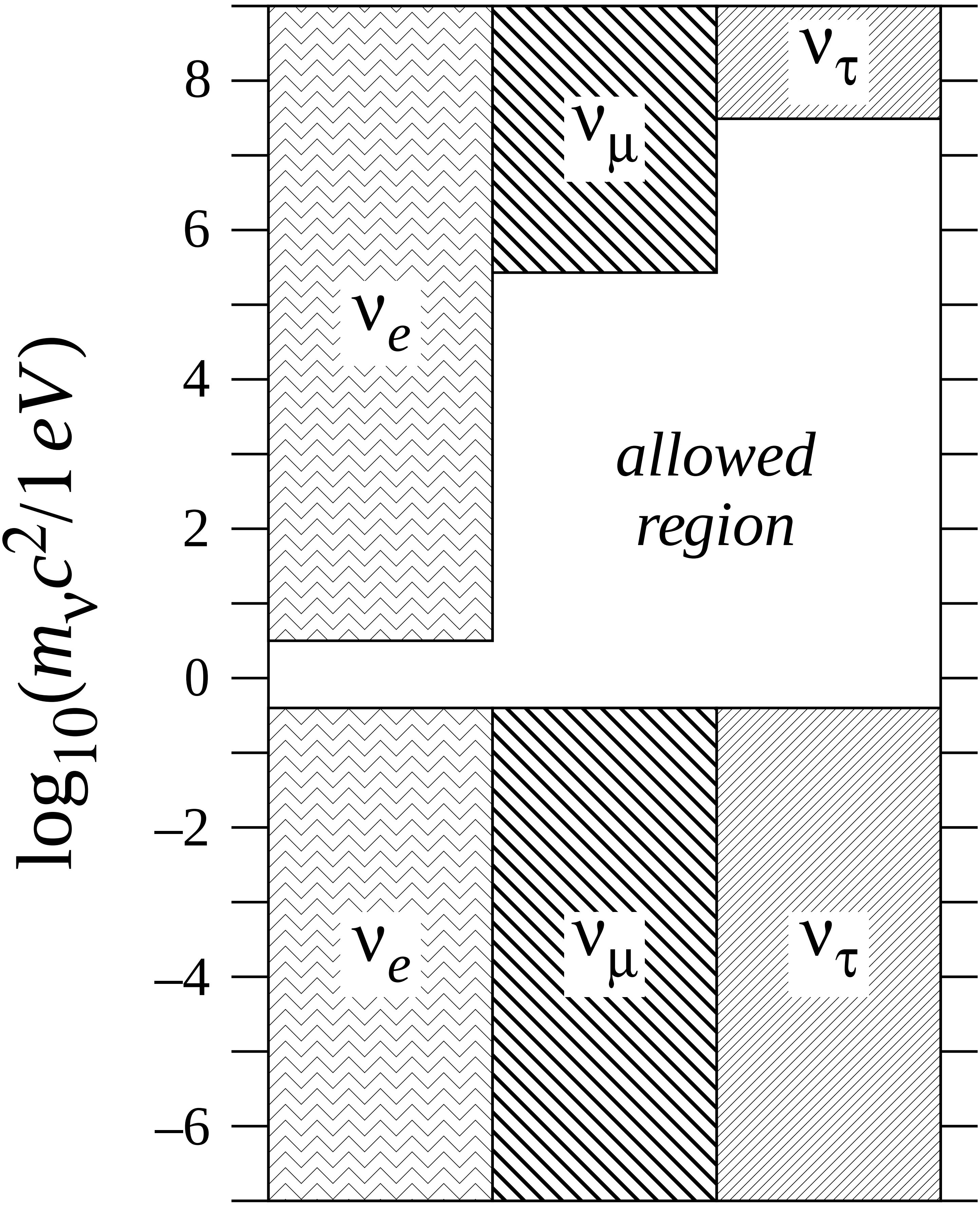

8
6
4
2
0
–2
–4
–6
$\log_{10}(m_\nu c^2/1\,eV)$
ν_τ
ν_μ
ν_e
allowed region
ν_e
ν_μ
ν_τ